

M 545 Homework Problem Solution

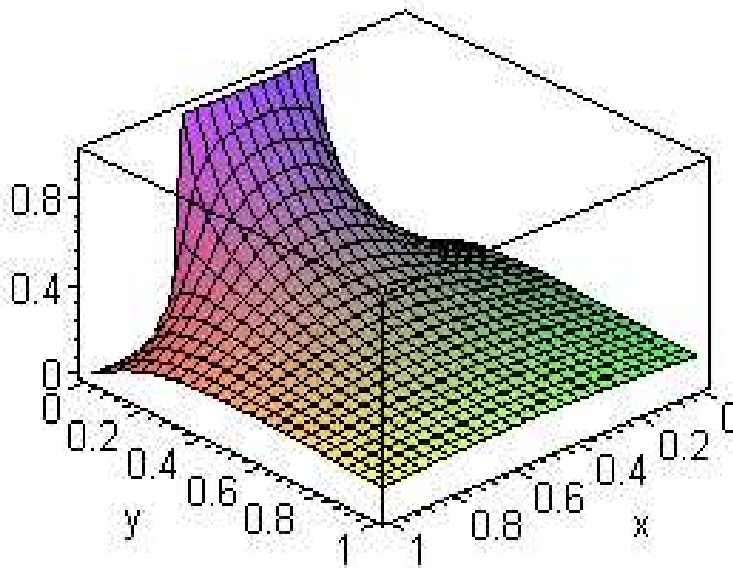
Suppose $u = u(x, y)$ solves

$$\begin{aligned} \nabla^2 u(x, y) &= 0 & x \in R, y > 0 \\ u(x, 0) &= I_{1/4}(x - 1/2) & x \in R \end{aligned}$$

the solution is:

$$\begin{aligned} u(x, y) &= T_F^{-1} \{ F(\alpha) e^{-y|\alpha|} \} \quad \text{note that you **do not** have to compute } F(\alpha) \\ &= \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(s)}{(x-s)^2 + y^2} ds \\ &= \frac{y}{\pi} \int_{1/4}^{3/4} \frac{1}{(x-s)^2 + y^2} ds = \frac{1}{\pi} \left[\arctan\left(\frac{x-1/4}{y}\right) - \arctan\left(\frac{x-3/4}{y}\right) \right] \end{aligned}$$

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u:=(x,y)->(y/Pi)*int(1/(y^2+(x-s)^2),s=1/4..3/4);
plot3d(u(x,y),x=0..1,y=0..1);
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Suppose $v = v(x, y)$ solves

$$\begin{aligned} \nabla^2 v(x, y) &= 0 & 0 < x < 1, 0 < y < 1 \\ v(x, 0) &= I_{1/4}(x - 1/2), & 0 < x < 1, \\ v(x, 1) &= 0 & , 0 < x < 1, \\ \partial_x v(0, y) &= \partial_x v(1, y) = 0 & 0 < y < 1. \end{aligned}$$

then the solution is:

$$v(x, y) = v_0(y) + \sum_{n=1}^{\infty} v_n(y) \cos(n\pi x)$$

where

$$v_0''(y) = 0 \quad v_0(0) = f_0, \quad v_0(1) = 0$$

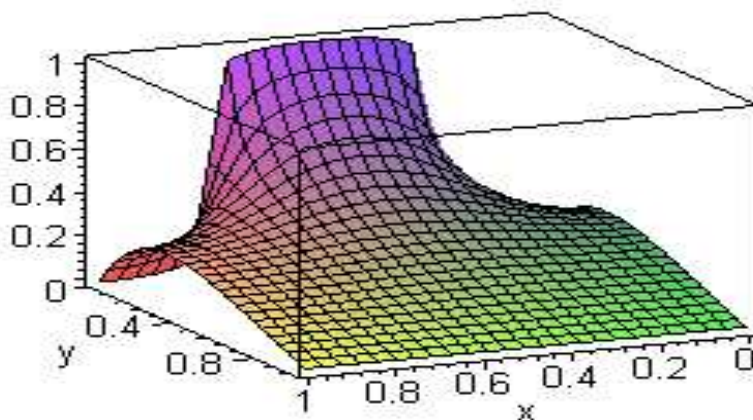
$$v_n''(y) = (n\pi)^2 v_n(y), \quad v_n(0) = f_n, \quad v_n(1) = 0$$

Then

$$v_0(y) = f_0(1-y) \quad v_n(y) = f_n \frac{\sinh(n\pi(1-y))}{\sinh(n\pi)}$$

i.e.,
$$v(x,y) = f_0(1-y) + \sum_{n=1}^{\infty} f_n \frac{\sinh(n\pi(1-y))}{\sinh(n\pi)} \cos(n\pi x)$$

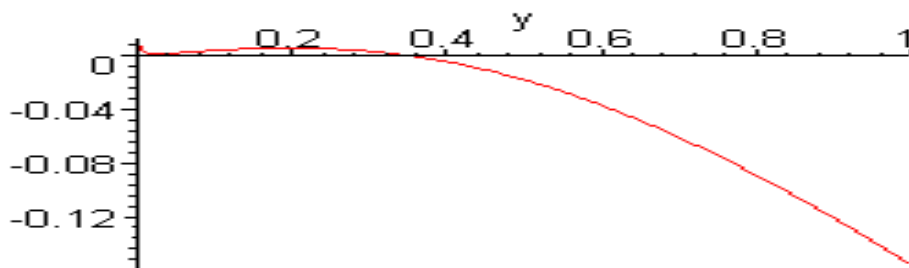
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f:=n->int(cos(n*Pi*x),x=1/4..3/4)/int(cos(n*Pi*x)^2,x=0..1);
v:=(x,y,N)->f(0)*(1-y)+sum(f(n)*sinh(n*Pi*(1-y))*cos(n*Pi*x)/sinh(n*Pi),n=1..N);
plot3d(v(x,y,50),x=0..1,y=0..1);
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1. Is $u(\frac{1}{2}, \frac{1}{2})$ less than, equal to, or larger than $v(\frac{1}{2}, \frac{1}{2})$?

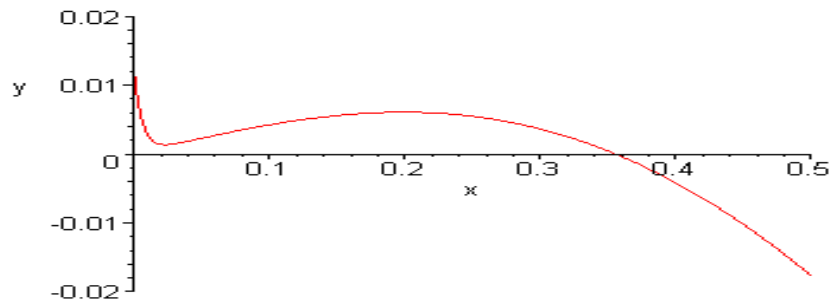
This is a plot of $v(\frac{1}{2}, y) - u(\frac{1}{2}, y)$ versus y from which it can be seen that

$$v(\frac{1}{2}, y) - u(\frac{1}{2}, y) < 0 \text{ at } y = \frac{1}{2}$$

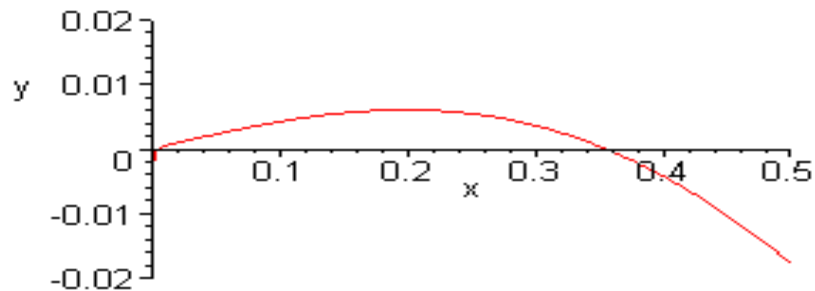


2. Does the difference $v(\frac{1}{2}, y) - u(\frac{1}{2}, y)$ change sign on $0 < y < 1$?

The plot of $v(\frac{1}{2}, y) - u(\frac{1}{2}, y)$ versus y shows one sign change on $0 < y < 1$. It also shows some odd behavior near $y = 0$, where the difference should be zero because of the boundary condition there. This plot was based on 50 terms in the sum for $v(x,y)$ and the difficulty may be due to that. To check, we redo the plot with $v(x,y)$ using 150 terms in the sum. That plot, shown below behaves much better near $y = 0$.



50 terms



150 terms

3. How would you explain your results in the first two questions? e.g. in the context of two heat conduction experiments what do your results mean?

The solutions $u(x,y)$ and $v(x,y)$ could represent the steady state temperature responses to identical square pulse boundary temperatures at $x = 0$. However, the heat conductor modeled by u is infinite in extent, whereas the conductor modeled by v is of finite size and has insulated lateral edges at $x = 0$ and $x = 1$ with the face opposite the heated face maintained at a constant temperature of zero.

Because the heat in the finite conductor is confined by the insulated faces at $x = 0$ and $x = 1$, the heat does not disperse as readily as does the heat in the unbounded conductor. This results $v(x,y)$ being larger than $u(x,y)$ for y close to zero (more precisely v is greater than u for $0 < y < 1/3$, approximately) As y increases away from zero, the fact that $v(x,1)$ is forced to be zero causes the temperature in the finite conductor to drop more rapidly than the temperature in the unbounded conductor, resulting in negative values for $v - u$ when y is closer to 1 than to 0.