M 545 Homework Assignment B   Solution

1. Suppose $u = u(x,y)$ solves

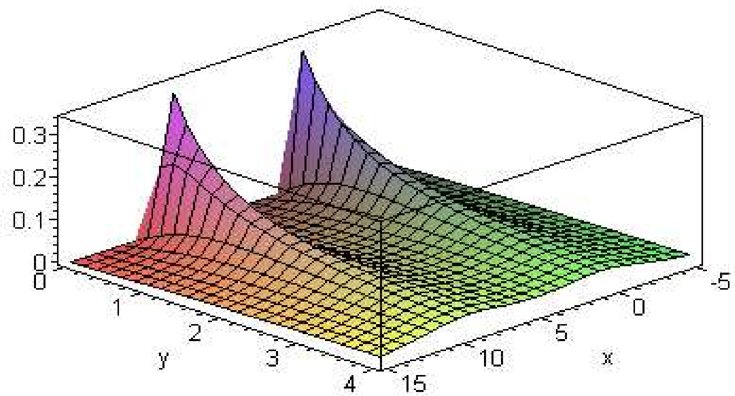
$$\nabla^2 u(x,y) = 0 \qquad x \in R,\ y > 0$$

$$u(x,0) = \begin{cases} x(1-x) & if \quad 0 < x < 1 \\ (x-9)(10-x) & if \quad 9 < x < 10 \\ 0 & otherwise \end{cases}$$

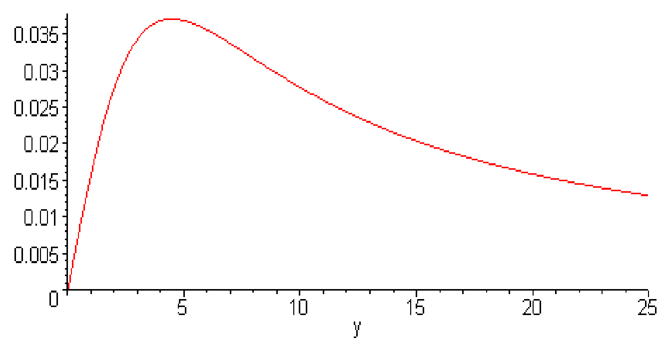
We know from past examples that

$$u(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{f(z)}{(x-z)^2 + y^2}dz$$
$$= \frac{y}{\pi}\int_0^1 \frac{z(1-z)}{(x-z)^2+y^2}dz + \frac{y}{\pi}\int_9^{10} \frac{(z-9)(10-z)}{(x-z)^2+y^2}dz$$

This solution looks like this:



To determine the values of y for which $u(5,y)$ is increasing and for which values it is decreasing, we plot $u(5,y)$



and see that it increases until about $y = 5$ and then decreases for $y > 5$.
The reason for the initial increase, explained in the context of heat conduction, is that heat flows away from the warm region around two pulses located in the intervals $(0,1)$ and $(9,10)$ toward regions of lower temperature. Most of the heat flows in the direction of

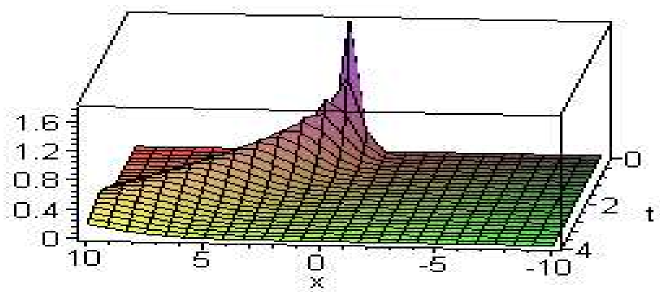
increasing y but for $0 < y < 5$, the flow in the x direction is sufficient to cause an increase in temperature in the region between the two pulses. For $y > 5$, the temperature pulses have dissipated sufficiently that there is no further increase in the region between the pulses. Note that by computing $-grad\, u(x,y)$ it is possible to see the direction of heat flow at any point (x,y) in the half-plane.

2. Suppose $u = u(x,t)$ solves

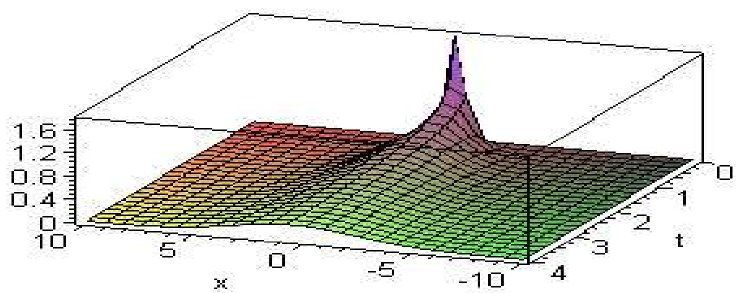
$$\partial_t u = D\,\partial_{xx} u + V\,\partial_x u \qquad x \in R,\ t > 0$$

$$u(x,0) = \begin{cases} 1 & \text{if} \quad 0 < x < 1 \\ 0 & \quad \text{otherwise} \end{cases}$$

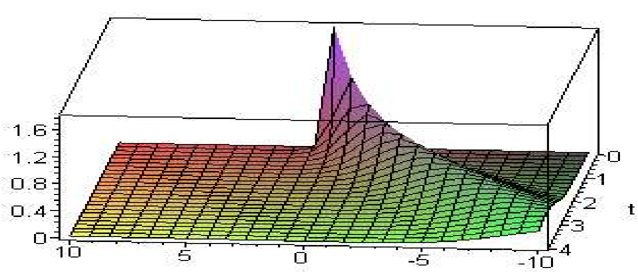
Find $u(x,t)$ and plot the solution for $D = 1$ for several values of V, both positive and negative.



$D = 1$ and $V = -3$



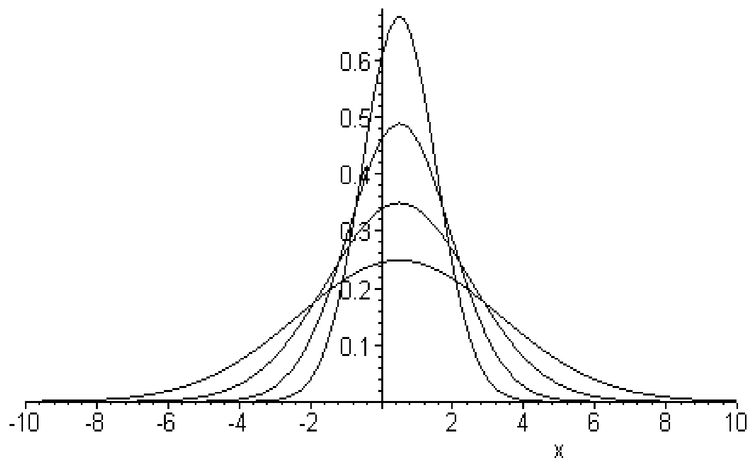
$D = 1$ and $V = 0$



$D = 1$ and $V = 3$

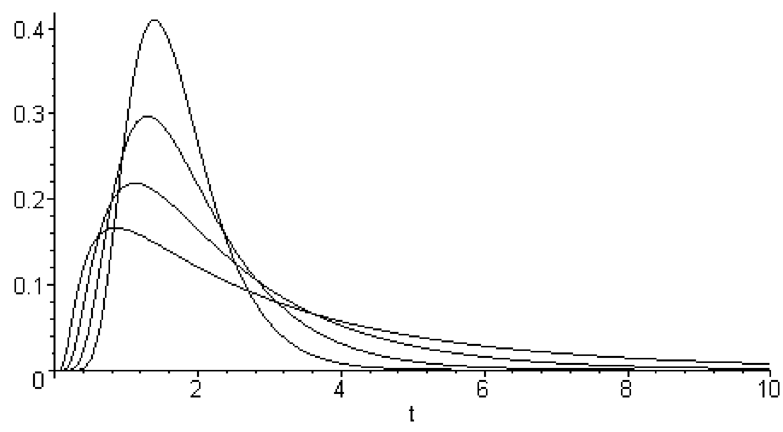
If we think of this as modeling a contaminant pulse which is propagated by both diffusion and convection then we can see that when $V = 0,$ the contaminant pulse remains centered about its initial location and simply spreads by diffusion. On the other hand, by examining the plots of u for $V \neq 0,$ we observe that V causes the solution to be convected in the positive x direction when $V < 0$ and convected in the negative x direction if $V > 0.$

For a fixed $V,$ plot the solution for several values of $D,$ (but only positive D). The first plot shows u versus x for a fixed t and $V = 0.$ It is evident that as D is increased, the diffusion progresses more rapidly.



$u(x,.5)$ for $V = 0$ and $D = 1,2,4,8$

The second plot shows u versus t at a fixed downstream value of x for several values of $D.$ It is evident from this plot that for small values of D, the contaminant pulse remains concentrated in a smaller area as it is convected downstream while for large values of $D,$ the pulse spreads out as it is convected.



$u(5,t)$ for $V = -3$ and $D = 1,2,4,8$

What is the influence of the parameters D and V on the solution to this problem?

3.Suppose $w = w(x,t)$ solves

$$\partial_t w = D\,\partial_{xx} w + V\,\partial_x w \qquad x \in (0,10),\ t > 0$$

$$u(x,0) = \begin{cases} 1 & if \quad 0 < x < 1 \\ 0 & otherwise \end{cases}$$

$$w(0,t) = w(10,t) = 0$$

Let $w(x,t) = e^{\alpha x} U(x,t)$ and find the equation satisfied by $U(x,t)$.

$$\partial_t w = e^{\alpha x} \partial_t U(x,t)$$
$$\partial_x w = e^{\alpha x}(\partial_x U(x,t) + \alpha U)$$
$$\partial_{xx} w = e^{\alpha x}(\partial_{xx} U(x,t) + 2\alpha \partial_x U + \alpha^2 U)$$

Then

$$e^{\alpha x}\partial_t U(x,t) = De^{\alpha x}(\partial_{xx}U(x,t) + 2\alpha\partial_x U + \alpha^2 U) + Ve^{\alpha x}(\partial_x U(x,t) + \alpha U)$$
$$\partial_t U(x,t) = D\partial_{xx}U(x,t) + (2\alpha D + V)\partial_x U + (\alpha^2 D + \alpha V)U$$

Next, choose α so that the equation for $U(x,t)$ contains no $\partial_x U$ term (but it will contain a U term)

If $\alpha = \frac{-V}{2D}$ *then* $\alpha^2 D + \alpha V = -\frac{V^2}{4D} = c$

and the equation for U contains no first derivative term,

$$\partial_t U(x,t) = D\partial_{xx}U(x,t) + cU$$

$$U(x,0) = \begin{cases} e^{-\alpha x} & if \quad 0 < x < 1 \\ 0 & otherwise \end{cases} = U_0(x)$$

$$U(0,t) = U(10,t) = 0$$

Identify the eigenfunctions for the U-problem

The S-L problem is $-X''(x) = \lambda X(x) \qquad X(0) = X(10) = 0$

hence $X_n(x) = \sin\left(\frac{n\pi x}{10}\right), \qquad \lambda_n = \left(\frac{n\pi x}{10}\right)^2 \quad n = 1,2,\ldots$

Write down the system of ODE's for the time dependent coefficients in the eigenfunction expansion of U,

$$U(x,t) = \sum_{n=1}^{\infty} U_n(t)\, X_n(x) \qquad U_0(x) = \sum_{n=1}^{\infty} a_n\, X_n(x)$$

where $U_n'(t) = (-\lambda_n D + c)U_n(t), \qquad U_n(0) = a_n = \frac{(U_0, X_n)}{(X_n, X_n)} = 2\int_0^1 e^{-\alpha x} \sin\left(\frac{n\pi x}{10}\right)dx$

Solve the ODE's to obtain $U(x,t)$

$$U_n(t) = a_n e^{(-\lambda_n D + c)t}$$

and

$$U(x,t) = \sum_{n=1}^{\infty} a_n e^{(-\lambda_n D + c)t}\, X_n(x)$$

Find the corresponding solution for $w(x,t)$

$$w(x,t) = e^{\alpha x + ct} \sum_{n=1}^{\infty} a_n e^{-\lambda_n D t} \sin\left(\frac{n\pi x}{10}\right)$$

$$\alpha = \frac{-V}{2D}, \qquad c = -\frac{V^2}{4D}$$