Solving PDE's Using the Fourier Transform

1. Consider the Neumann problem

$$\begin{aligned} \nabla^2 u(x,y) &= 0, \qquad x \in R, \quad y > 0, \\ \partial_y u(x,0) &= g(x), \;\; x \in R. \end{aligned}$$

(a) Let $v(x,y) = \partial_y u(x,y)$ and show that $v(x,y)$ then solves the Dirichlet problem,

$$\begin{aligned} \nabla^2 v(x,y) &= 0, \qquad x \in R, \quad y > 0, \\ v(x,0) &= g(x), \;\; x \in R. \end{aligned}$$

(b) Show that $$v(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{y^2 + (x-z)^2} g(z)dz = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{y^2 + z^2} g(x-z)dz$$

(c) Show that $$\partial_y\left[\tfrac{1}{2}\ln(y^2+z^2)\right] = \frac{y}{y^2+z^2}$$

so $$\partial_y u(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \partial_y\left[\tfrac{1}{2}\ln(y^2+z^2)\right] g(x-z)dz$$

and $$u(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \ln(y^2+z^2) g(x-z)dz = \frac{1}{2\pi}\int_{-\infty}^{\infty} \ln\left(y^2+(x-z)^2\right) g(z)dz$$

(d) Is this solution unique?

2. Consider the problem

$$\begin{aligned} \partial_t u(x,t) &= K\partial_{xx}u(x,t) + V\partial_x u(x,t) + b\,u(x,t), \quad x \in R, \quad t > 0, \\ u(x,0) &= f(x), \qquad x \in R. \end{aligned}$$

Denote the solution of this problem by $u(x,t;K,V,b)$.

(a) Show that $U(\alpha,t) = T_F[u(x,t)]$ is given by

$$U(\alpha,t) = F(\alpha)\,e^{-Kt\alpha^2}\; e^{iVt\alpha}\; e^{bt}$$

(b) Show that $$T_F^{-1}\left[F(\alpha)\,e^{-Kt\alpha^2}\right] = \frac{1}{\sqrt{4\pi Kt}}\int_{-\infty}^{\infty} e^{-\frac{(x-z)^2}{4Kt}} f(z)dz = u(x,t;K,0,0)$$

(c) Show that $$T_F^{-1}\left[F(\alpha)\,e^{-Kt\alpha^2} e^{iVt\alpha}\right] = u(x+Vt,t;K,0,0) = u(x,t;K,V,0)$$

(d) Show that $$T_F^{-1}\left[F(\alpha)\,e^{-Kt\alpha^2} e^{iVt\alpha}\, e^{bt}\right] = e^{bt}\,u(x+Vt,t;K,0,0) = u(x,t;K,V,b)$$

3. Consider the boundary value problem on a quarter plane,

$$\begin{aligned} \nabla^2 u(x,y) &= 0, \qquad x > 0, \quad y > 0, \\ u(x,0) &= f(x), \;\; x > 0, \\ u(0,y) &= 0, \qquad y > 0, \end{aligned}$$

and let $v(x,y)$ solve $$\begin{aligned} \nabla^2 v(x,y) &= 0, \qquad x \in R, \quad y > 0, \\ v(x,0) &= \hat{f}(x), \;\; x \in R, \end{aligned}$$

where

$$\hat{f}(x) = \begin{cases} f(x) & if \quad x > 0 \\ -f(-x) & if \quad x < 0 \end{cases} \quad \text{(odd extension of } f(x) \text{)}$$

(a) Show that

$$\begin{aligned} v(x,y) &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{y^2+(x-z)^2}\hat{f}(z)dz \\ &= \frac{1}{\pi}\int_{-\infty}^{0} \frac{-y}{y^2+(x-z)^2}f(-z)dz + \frac{1}{\pi}\int_{0}^{\infty} \frac{y}{y^2+(x-z)^2}f(z)dz \\ &= \frac{1}{\pi}\int_{0}^{\infty} \frac{-y}{y^2+(x+z)^2}f(z)dz + \frac{1}{\pi}\int_{0}^{\infty} \frac{y}{y^2+(x-z)^2}f(z)dz \\ &= \frac{1}{\pi}\int_{0}^{\infty}\left[\frac{y}{y^2+(x-z)^2} - \frac{y}{y^2+(x+z)^2} \right]f(z)dz, \end{aligned}$$

and so $v(0,y) = 0.$

(b) Show that $v(x,y)$ satisfies

$$\begin{aligned} \nabla^2 v(x,y) &= 0, \quad x > 0, \quad y > 0, \\ v(x,0) &= f(x), \quad x > 0, \\ v(0,y) &= 0, \quad y > 0, \end{aligned}$$

and therefore, by uniqueness, it must be the case that $u(x,y) = v(x,y) \quad for \quad x > 0, \quad y > 0.$

4. Consider the boundary value problem on a quarter plane,

$$\begin{aligned} \nabla^2 u(x,y) &= 0, \quad x > 0, \quad y > 0, \\ u(x,0) &= f(x), \quad x > 0, \\ \partial_x u(0,y) &= 0, \quad y > 0, \end{aligned}$$

and let $v(x,y)$ solve

$$\begin{aligned} \nabla^2 v(x,y) &= 0, \quad x \in R, \quad y > 0, \\ v(x,0) &= \hat{f}(x), \quad x \in R, \end{aligned}$$

where

$$\hat{f}(x) = \begin{cases} f(x) & if \quad x > 0 \\ f(-x) & if \quad x < 0 \end{cases} \quad \text{(even extension of } f(x) \text{)}$$

(a) Show that

$$\begin{aligned} v(x,y) &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{y^2+(x-z)^2}\hat{f}(z)dz \\ &= \frac{1}{\pi}\int_{-\infty}^{0} \frac{y}{y^2+(x-z)^2}f(-z)dz + \frac{1}{\pi}\int_{0}^{\infty} \frac{y}{y^2+(x-z)^2}f(z)dz \\ &= \frac{1}{\pi}\int_{0}^{\infty} \frac{y}{y^2+(x+z)^2}f(z)dz + \frac{1}{\pi}\int_{0}^{\infty} \frac{y}{y^2+(x-z)^2}f(z)dz \\ &= \frac{1}{\pi}\int_{0}^{\infty}\left[\frac{y}{y^2+(x-z)^2} + \frac{y}{y^2+(x+z)^2} \right]f(z)dz, \end{aligned}$$

and so $\partial_x v(0,y) = 0.$

(b) Show that $v(x,y)$ satisfies

$$\begin{aligned} \nabla^2 v(x,y) &= 0, \quad x > 0, \quad y > 0, \\ v(x,0) &= f(x), \quad x > 0, \\ \partial_x v(0,y) &= 0, \quad y > 0, \end{aligned}$$

and therefore, by uniqueness, it must be the case that $u(x,y) = v(x,y) \quad \mathit{for}$ $x > 0, \quad y > 0.$

5.Consider the boundary value problem on a quarter plane,

$$\begin{aligned} \nabla^2 u(x,y) &= 0, \quad x > 0, \quad y > 0, \\ \partial_y u(x,0) &= f(x), \quad x > 0, \\ u(0,y) &= 0, \quad y > 0, \end{aligned}$$

and let $v(x,y)$ solve

$$\begin{aligned} \nabla^2 v(x,y) &= 0, \quad x \in R, \quad y > 0, \\ v(x,0) &= \hat{f}(x), \quad x \in R, \end{aligned}$$

where

$$\hat{f}(x) = \begin{cases} f(x) & \mathit{if} \quad x > 0 \\ -f(-x) & \mathit{if} \quad x < 0 \end{cases}$$ (odd extension of $f(x)$)

(a) Show that

$$\begin{aligned} v(x,y) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \ln\left(y^2 + (x-z)^2\right)\hat{f}(z)dz \\ &= \frac{-1}{2\pi}\int_{-\infty}^{0} \ln\left(y^2 + (x-z)^2\right)f(-z)dz + \frac{1}{2\pi}\int_{0}^{\infty} \ln\left(y^2 + (x-z)^2\right)f(z)dz \\ &= \frac{-1}{2\pi}\int_{0}^{\infty} \ln\left(y^2 + (x+z)^2\right)f(z)dz + \frac{1}{2\pi}\int_{0}^{\infty} \ln\left(y^2 + (x-z)^2\right)f(z)dz \\ &= \frac{1}{2\pi}\int_{0}^{\infty} \left[\ln\left(y^2 + (x-z)^2\right) - \ln\left(y^2 + (x+z)^2\right)\right]f(z)dz, \\ &= \frac{1}{2\pi}\int_{0}^{\infty} \left[\ln\frac{y^2 + (x-z)^2}{y^2 + (x+z)^2}\right]f(z)dz, \end{aligned}$$

and so $v(0,y) = 0.$

(b) Show that $v(x,y)$ satisfies

$$\begin{aligned} \nabla^2 v(x,y) &= 0, \quad x > 0, \quad y > 0, \\ \partial_y v(x,0) &= f(x), \quad x > 0, \\ v(0,y) &= 0, \quad y > 0, \end{aligned}$$

and therefore, by uniqueness, it must be the case that $u(x,y) = v(x,y) \quad \mathit{for}$ $x > 0, \quad y > 0.$

6.Consider the boundary value problem on a quarter plane,

$$\begin{aligned} \nabla^2 u(x,y) &= 0, \quad x > 0, \quad y > 0, \\ \partial_y u(x,0) &= f(x), \quad x > 0, \\ \partial_x u(0,y) &= 0, \quad y > 0, \end{aligned}$$

and let $v(x,y)$ solve
$$\begin{aligned}\nabla^2 v(x,y) &= 0, \quad x \in R, \quad y > 0,\\ v(x,0) &= \hat{f}(x), \quad x \in R,\end{aligned}$$

where

$$\hat{f}(x) = \begin{cases} f(x) & if \quad x > 0 \\ f(-x) & if \quad x < 0 \end{cases}$$ (even extension of $f(x)$)

(a) Show that
$$\begin{aligned} v(x,y) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \ln\left(y^2 + (z-x)^2\right)\hat{f}(z)dz \\ &= \frac{1}{2\pi}\int_{-\infty}^{0} \ln\left(y^2 + (x-z)^2\right)f(-z)dz + \frac{1}{2\pi}\int_{0}^{\infty} \ln\left(y^2 + (z-x)^2\right)f(z)dz \\ &= \frac{1}{2\pi}\int_{0}^{\infty} \ln\left(y^2 + (x+z)^2\right)f(z)dz + \frac{1}{2\pi}\int_{0}^{\infty} \ln\left(y^2 + (z-x)^2\right)f(z)dz \\ &= \frac{1}{2\pi}\int_{0}^{\infty}\left[\ln\left(y^2 + (z-x)^2\right) + \ln\left(y^2 + (x+z)^2\right)\right]f(z)dz, \\ &= \frac{1}{2\pi}\int_{0}^{\infty}\ln\left[\left(y^2 + (z-x)^2\right)\left(y^2 + (x+z)^2\right)\right]f(z)dz, \end{aligned}$$

and so $\partial_x v(0,y) = 0.$

(b) Show that $v(x,y)$ satisfies
$$\begin{aligned}\nabla^2 v(x,y) &= 0, \quad x > 0, \quad y > 0,\\ \partial_y v(x,0) &= f(x), \quad x > 0,\\ \partial_x v(0,y) &= 0, \quad y > 0,\end{aligned}$$

and therefore, by uniqueness, it must be the case that $u(x,y) = v(x,y) \quad for$ $x > 0, \quad y > 0.$

7.Consider the boundary value problem on a quarter plane,

$$\begin{aligned}\nabla^2 u(x,y) &= 0, \quad x > 0, \quad y > 0,\\ \partial_y u(x,0) &= f(x), \quad x > 0,\\ u(0,y) &= g(y), \quad y > 0.\end{aligned}$$

(a) Show that $u(x,y) = v(x,y) + w(x,y) \quad where$

$$\begin{aligned}\nabla^2 v(x,y) &= 0, \quad x > 0, \quad y > 0,\\ \partial_y v(x,0) &= f(x), \quad x > 0,\\ v(0,y) &= 0, \quad y > 0,\end{aligned} \quad and \quad \begin{aligned}\nabla^2 w(x,y) &= 0, \quad x > 0, \quad y > 0,\\ \partial_y w(x,0) &= 0, \quad x > 0,\\ w(0,y) &= g(y), \quad y > 0.\end{aligned}$$

(b) Refer to previous problems for the solutions to the v and w problems and write down the solution for u.

8.Consider the boundary value problem on a quarter plane,

$$\begin{aligned}\nabla^2 u(x,y) &= 0, \quad x > 0, \quad y > 0,\\ u(x,0) &= f(x), \quad x > 0,\\ \partial_x u(0,y) &= g(y), \quad y > 0.\end{aligned}$$

(a) Show that $u(x,y) = v(x,y) + w(x,y) \quad where$

$$\begin{aligned} \nabla^2 v(x,y) &= 0, \quad x > 0, \quad y > 0, \quad and \qquad & \nabla^2 w(x,y) &= 0, \quad x > 0, \quad y > 0, \\ v(x,0) &= f(x), \quad x > 0, & w(x,0) &= 0, \quad x > 0, \\ \partial_x v(0,y) &= 0, \quad y > 0, & \partial_x w(0,y) &= g(y), \quad y > 0. \end{aligned}$$

(b) Refer to previous problems for the solutions to the v and w problems and write down the solution for u.

9. Consider the boundary value problem for Laplace's equation on an infinite strip,

$$\begin{aligned} \nabla^2 u(x,y) &= 0, \quad x \in R, \quad 0 < y < 1, \\ u(x,0) &= 0, \quad x \in R, \\ u(x,1) &= f(x), \quad x \in R. \end{aligned}$$

(a) Show that $U(\alpha,y) = T_F[u(x,y)]$ can be expressed as

$$U(\alpha,y) = F(\alpha)\frac{e^{y|\alpha|} + e^{-y|\alpha|}}{e^{|\alpha|} - e^{-|\alpha|}}$$

(b) Use the fact that

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad if \qquad |x| < 1$$

to write

$$U(\alpha,y) = F(\alpha)\sum_{n=0}^{\infty}(-1)^n e^{-(2n+1-y)|\alpha|} + F(\alpha)\sum_{n=0}^{\infty}(-1)^n e^{-(2n+1+y)|\alpha|}$$

(c) Find $u(x,y)$.

10. Consider the boundary value problem for Laplace's equation on a semi-infinite strip,

$$\begin{aligned} \nabla^2 u(x,y) &= 0, \quad x > 0, \quad 0 < y < 1, \\ \partial_x u(0,y) &= 0, \quad 0 < y < 1 \\ u(x,0) &= 0, \quad x > 0, \\ u(x,1) &= f(x), \quad x > 0. \end{aligned}$$

(a) Use an even or odd extension procedure to convert this problem to a problem on an infinite strip.that you have solved before.

(b) Use the previously found solution for the infinite strip problem to express the solution to the problem on the semi-infinite strip writing it in terms of the extended data function, $\tilde{f}$

(c) Express the solution to the problem on the semi-infinite strip in terms of f, not the extension of f.