Solving PDE’s by Eigenfunction Expansion

Some of these problems are difficult and you should ask questions (either after class or in my office) to help you get started and after starting, to make sure you are proceeding correctly.

1. Consider the initial value problem for the heat equation

\[ \partial_t u(x, t) = D \partial_x u(x, t), \quad 0 < x < 1, \ t > 0, \]

\[ u(x, 0) = f(x) \in L_2(0, 1) \]

with BC’s

(i) \( u(0, t) = u(1, t) = 0 \)
(ii) \( \partial_x u(0, t) = u(1, t) = 0 \)
(iii) \( \partial_x u(0, t) = \partial_x u(1, t) = 0 \)

a) Find the eigenvalues and eigenfunctions in each of the 3 cases
b) Try to determine which problem has the solution reaching the steady state soonest.

hint: The more negative the eigenvalues, the faster the components of the solution decay to zero. So, look at the smallest eigenvalue in each case and note that this term dictates the decay to zero.

Are the steady states the same in all 3 cases?

2. Consider the initial value problem for the heat equation with convection,

\[ \partial_t u(x, t) = D \partial_x u(x, t) + V \partial_x u(x, t), \quad 0 < x < 1, \ t > 0, \]

\[ u(x, 0) = f(x) \in L_2(0, 1) \]

\[ u(0, t) = u(1, t) = 0. \]

a) Let

\[ u(x, t) = e^{ax + bt}w(x, t) \]

and show

\[ \partial_t u = e^{ax + bt}[\partial_t w(x, t) + bw] \]

\[ \partial_x u = e^{ax + bt}[\partial_x w(x, t) + aw] \]

\[ \partial_{xx} u = e^{ax + bt} [\partial_{xx} w(x, t) + 2a \partial_x w + a^2 w] \]

b) show

\[ \partial_t w(x, t) + bw = D[\partial_{xx} w(x, t) + 2a \partial_x w + a^2 w] \]

\[ + V[\partial_x w(x, t) + aw] \]

c) choose the parameters \( a \) and \( b \) such that \( w(x, t) \) solves

\[ \partial_t w(x, t) = D \partial_{xx} w(x, t) \]

\[ w(x, 0) = e^{-ax}f(x) \in L_2(0, 1) \]

\[ w(0, t) = w(1, t) = 0. \]

d) Solve for \( w \) and then find \( u(x, t) \). How does the solution when \( V \neq 0 \) differ from the solution when \( V = 0 \)?

3. Consider the wave equation with dispersion,

\[ \partial_{tt} u(x, t) = a^2 \partial_{xx} u(x, t) - d^2 \partial_t u(x, t), \quad 0 < x < 1, \ t > 0, \]

\[ u(x, 0) = f(x) \in L_2(0, 1) \]

\[ \partial_t u(x, 0) = 0, \]

\[ u(0, t) = u(1, t) = 0. \]

a) expand the solution in series of the form

\[ u(x, t) = \sum_{n=1}^{\infty} u_n(t) \phi_n(x) \]
and find the ODE problem solved by $u_n(t)$.

b) Find $u_n(t)$ and $u(x,t)$.

c) Compare $u(x,t)$ to the solution obtained in the case $d = 0$, which is

$$u(x,t) = \sum_{n=1}^{\infty} f_n \cos(n\pi at) \sin(n\pi x)$$

$$= \frac{1}{2} \sum_{n=1}^{\infty} f_n \left[ \sin(n\pi(x + at)) + \sin(n\pi(x - at)) \right]$$

$$= \frac{1}{2} \left[ \tilde{f}(x + at) + \tilde{f}(x - at) \right]$$

where $\tilde{f}$ denotes a periodic extension of the function $f$, defined on $[0, 1]$.

4. Suppose $u(x,t)$ solves:

$$\partial_t u(x,t) = \partial_{xx} u(x,t), \quad 0 < x < 1, \quad t > 0$$

$$u(x,0) = 0, \quad 0 < x < 1,$$

$$u(0,t) = T_0, \quad u(1,t) = T_1, \quad t > 0.$$

(a) Use the change of dependent variable

$$v(x,t) = u(x,t) - T_0(1-x) - T_1 x$$

...to convert this to a problem in $v(x,t)$ with homogeneous boundary conditions

(b) Solve the problem for $v(x,t)$ and use the solution to give an expression for $u(x,t)$.

(c) Explain the statement that $v(x,t)$ represents the transient part of the response, $u(x,t)$, and identify the steady state part of $u(x,t)$.

5. Consider the Sturm-Liouville problem:

$$-X''(x) = \lambda X(x), \quad 0 < x < 1,$$

$$X'(0) + aX(0) = 0$$

$$X'(1) + bX(1) = 0$$

In each of the following cases, determine all the eigenvalues and eigenfunctions:

(a) $a = 0, \quad b = 1$

(b) $a = 6, \quad b = 3$

Hint: (i) if $\lambda = -\mu^2 < 0$ then $X(x) = A \cosh \mu x + B \sinh \mu x$, $X'(x) = \mu A \sinh \mu x + \mu B \cosh \mu x$, and the boundary conditions imply

$$A = -(\mu/a)B,$$

$$\tanh \mu = \frac{\mu(b-a)}{\mu^2 - ab}$$

With $a = 6, \quad b = 3$, plot $\tanh \mu$, and $\frac{\mu(b-a)}{\mu^2 - ab}$ vs $\mu$ on the same axes:
One nonzero root near \( \mu = 6 \) provides a negative e-value of \( \lambda_{-1} \approx -36 \)

(ii) if \( \lambda = 0 \) then \( X(x) = Ax + B \) and the boundary conditions imply \( A = B = 0 \) provided \( a - b + ab \neq 0 \)

(iii) if \( \lambda = \mu^2 > 0 \) then \( X(x) = A \cos \mu x + B \sin \mu x, \quad X'(x) = -\mu A \sin \mu x + \mu B \cos \mu x, \)

and the boundary conditions imply

\[
A = -\left(\frac{\mu}{a}\right)B, \quad \tan \mu = \frac{\mu(a-b)}{\mu^2+ab}
\]

Plot \( \tan \mu \), and \( \frac{\mu(a-b)}{\mu^2+ab} \) vs \( \mu \) on the same axes:

infinitely many nonzero roots \( \mu_n \)

There are an infinite number of nonzero roots \( \mu_n \) and as \( n \) increases, \( \mu_n \) approaches \( n\pi \).

Use the same approach for the cases:

(c) \( a = 3, \ b = 6 \)

(d) \( a = 4, \ b = -16 \)

6. A Radon source is placed at one end of a thin, air-filled tube of length \( L \). The opposite end of the tube is sealed against the passage of Radon and a Radon detector is placed at the sealed end. Initially the tube is free of Radon, and as Radon diffuses down the tube, some of the Radon escapes through the sides of the tube due to leakage. This situation can be modelled by

\[
\begin{align*}
\frac{\partial}{\partial t} C(x,t) &= D \frac{\partial^2}{\partial x^2} C(x,t) - \beta C(x,t), \quad 0 < x < L, \ t > 0, \\
C(x,0) &= 0, \quad 0 < x < L,
\end{align*}
\]
\[
C(0,t) = C_0 > 0, \quad t > 0, \\
\partial_t C(L,t) = 0, \quad t > 0. 
\]

Here \( C(x,t) \) denotes the concentration of Radon at \((x,t)\), \( D \) denotes the diffusivity of Radon in air, and \( \beta \) denotes a positive constant related to the Radon leakage.

(a) Find \( C(L,t) \) for \( t > 0 \).

(b) How does the limit of \( C(L,t) \) as \( t \) tends to infinity in the case \( \beta > 0 \) differ from the limit in the case \( \beta = 0 \)?

7. Consider the problem:

\[
\partial_{xx} u(x,y) + \partial_{yy} u(x,y) = 0, \quad 0 < x,y < 1 \\
u(0,y) = f(y), \quad 0 < y < 1 \\
\partial_y u(1,y) = 0, \quad 0 < y < 1 \\
u(x,0) = 0, \quad 0 < x < 1 \\
u(x,1) = g(x), \quad 0 < x < 1 
\]

(a) Split this into two subproblems and identify the eigenfunctions for each problem. Why is it necessary to split the problem in two parts?

(b) Solve each of the subproblems and explain why \( u(x,y) \) equals the sum of the two subproblem solutions.

(c) If \( f(1) \neq g(0) \) then \( u(0,1) \) must assume two different values at the same point. Does the solution \( u(x,y) \) really do this, and if so, how?