Extra Problems for Laplace Equation

1. Suppose $u \in C^2(U) \cap C(\bar{U})$ satisfies

$$\nabla^2 u(\vec{x}) = F(x) \quad \text{for } x \in U,$$

$$\alpha u(x) + \beta \partial_N u(x) = g(x) \quad \text{for } x \in \partial U$$

Show that if the constants, α and β have the same sign, then this problem has at most one solution.
Does this follow if α and β do not have the same sign?

2. Show that there exists at most one $u(x) \in C^2(U) \cap C^1(\bar{U})$ solving

$$-\nabla^2 u(\vec{x}) + u(x) = F(x), \quad x \in U,$$

$$\partial_N u(x) = g \qquad \text{for } x \in \partial U$$

3. Show that there exists at most one $u(x) \in C^2(U) \cap C^1(\bar{U})$ solving

$$-\nabla^2 u(\vec{x}) + u(x) = F(x), \ x \in U,$$

$$u(x) = f_1(x) \quad \text{for } x \in \partial U_1$$

$$\partial_N u(x) = 0 \quad \text{for } x \in \partial U_{2.}$$

4. Show that there exists no $u(x) \in C^2(U) \cap C^1(\bar{U})$ solving

$$-\nabla^2 u(\vec{x}) + c(x)\, u(x) = F(x), \ x \in U,$$

$$\alpha u(x) + \beta \partial_N u(x) = 0 \quad \text{for } x \in \partial U$$

unless $F(x)$ satisfies the condition,

$$\int_U F(x)\, v(x)\, dx = 0 \qquad \text{for all } v = v(x) \in C^2(U) \cap C^1(\bar{U})$$

such that,

$$-\nabla^2 v(\vec{x}) + c(x)\, v(x) = 0, \ x \in U,$$

$$\alpha v(x) + \beta \partial_N v(x) = 0 \quad \text{for } x \in \partial U.$$

5. Suppose $u, v \in C^2(U) \cap C(\bar{U})$ satisfy

$$-\nabla^2 u(\vec{x}) = F(x) \quad \text{for } x \in U,$$

$$u(x) = g(x) \quad \text{for } x \in \partial U$$

and

$$-\nabla^2 v(\vec{x}) = F(x) + \varepsilon |x|^2 \quad \text{for } x \in U,$$

$$v(x) = g(x) \quad for\ x \in \partial U.$$

Use the max-min principle to show that if $\varepsilon > 0$ then $v(x) \geq u(x) \quad for\ x \in \bar{U}.$

6. Suppose $u \in C^2(U) \cap C(\bar{U})$ satisfies

$$\nabla^2 u(\vec{x}) = F(x) \quad for\ x \in U,$$

$$u(x) + \partial_N u(x) = g(x) \quad for\ x \in \partial U\ .$$

Then express $u(x)$ in terms of a Green's function $G(x,y) = E(x-y) - \phi(x,y);$ i.e., start with the symmetric Green's identity and find the boundary value problem that $\phi(x,y)$ must satisfy in order to express u in terms of integrals involving the Green's function and the data F and g.

7. Suppose $u \in C^2(U) \cap C(\bar{U})$ satisfies

$$\nabla^2 u(\vec{x}) = F(x) \qquad for\ x \in U,$$

$$u(x) = g_1(x) \quad for\ x \in \partial U_1$$

$$\partial_N u(x) = g_2(x) \quad for\ x \in \partial U_2\ .$$

Then express $u(x)$ in terms of a Green's function $G(x,y) = E(x-y) - \phi(x,y);$ i.e., start with the symmetric Green's identity and find the boundary value problem that $\phi(x,y)$ must satisfy in order to express u in terms of integrals involving the Green's function and the data F and $g_1\ and\ g_2.$