

# The Conduction-Diffusion Equation

## 1. Diffusion

Consider a region  $U$  in  $R^n$  containing a contaminant which diffuses through some diffusion medium which fills  $U$ . Let  $u(x, t)$  denote the scalar concentration of contaminant at position  $x$  in  $U$  at time  $t$ . Let  $F(x, t)$  denote the vector contaminant flux at position  $x$  in  $U$  at time  $t$ . Then for any open ball  $B$  in  $U$  and for any time interval  $(t_1, t_2)$  conservation of contaminant requires that

$$\int_B u(\vec{x}, t_2) dx = \int_B u(\vec{x}, t_1) dx - \int_{t_1}^{t_2} \int_{\partial B} F(\vec{x}, t) \cdot n(\vec{x}) dS(\vec{x}) dt$$

In the usual way, we obtain the integral form of the conservation equation

$$\int_{t_1}^{t_2} \int_B [\partial_t u(\vec{x}, t) + \text{div} F(\vec{x}, t)] dx dt = 0 \quad \text{for all } B \subset U, \text{ and all } (t_1, t_2). \quad (1.1)$$

Since the integrand here is assumed to be continuous, it follows that

$$\partial_t u(\vec{x}, t) + \text{div} F(\vec{x}, t) = 0, \quad \text{for all } \vec{x} \in U, \text{ and all } t \quad (1.2)$$

In the phenomenon of diffusion, contaminant flows from regions of high concentration to regions of lower concentration at a rate which is proportional to the rate of change of concentration. We can express this mathematically by writing

$$F(\vec{x}, t) = -D \text{grad} u(\vec{x}, t) \quad (1.3)$$

where the positive proportionality factor,  $D$  (diffusivity), will be assumed here to be constant. Using (1.3) in (1.2) leads to

$$\partial_t u(\vec{x}, t) - D \text{div}(\text{grad} u(\vec{x}, t)) = 0, \quad \text{for all } \vec{x} \in U, \text{ and all } t \quad (1.4)$$

This is the partial differential equation, sometimes called the **diffusion equation**, which governs the movement of contaminant by the mechanism of diffusion.

## 2. Heat Conduction

Consider a region  $U$  in  $R^n$  containing a heat conducting medium and let  $u(x, t)$  denote the temperature at position  $x$  in  $U$  at time  $t$ . Let  $F(x, t)$  denote the vector heat flux at position  $x$  in  $U$  at time  $t$ . Then for any open ball  $B$  in  $U$  and for any time interval  $(t_1, t_2)$ ,

$$\Delta Q = \int_B \rho C u(\vec{x}, t_2) dx - \int_B \rho C u(\vec{x}, t_1) dx$$

equals the net change in the amount of heat present in the ball  $B$  during the time interval  $(t_1, t_2)$ . Here  $\rho$  denotes the mass density and  $C$  denotes the material dependent parameter known as specific heat. Then conservation of heat requires that

$$\int_B \rho C u(\vec{x}, t_2) dx - \int_B \rho C u(\vec{x}, t_1) dx = - \int_{t_1}^{t_2} \int_{\partial B} F(\vec{x}, t) \cdot n(\vec{x}) dS(\vec{x}) dt$$

In the usual way, we obtain the integral and differential forms of the conservation of heat equation

$$\int_{t_1}^{t_2} \int_B [\rho C \partial_t u(\vec{x}, t) + \operatorname{div} F(\vec{x}, t)] dx dt = 0 \quad \text{for all } B \subset U, \text{ and all } (t_1, t_2). \quad (2.1)$$

and,

$$\rho C \partial_t u(\vec{x}, t) + \operatorname{div} F(\vec{x}, t) = 0, \quad \text{for all } \vec{x} \in U, \text{ and all } t \quad (2.2)$$

The conduction of heat proceeds such that heat flows from regions of high temperature to regions of lower temperature at a rate proportional to the temperature gradient (i.e., at a rate proportional to the spatial rate of change of temperature). This is expressed by

$$F(\vec{x}, t) = -K \operatorname{grad} u(\vec{x}, t) \quad (2.3)$$

where the positive proportionality factor is called thermal conductivity in the conduction setting. Then the partial differential equation of heat conduction becomes

$$\rho C \partial_t u(\vec{x}, t) - K \operatorname{div}(\operatorname{grad} u(\vec{x}, t)) = 0, \quad \text{for all } \vec{x} \in U, \text{ and all } t \quad (2.4)$$

Evidently, while the physical phenomena of conduction and diffusion are quite different, the mathematical models are identical. In discussing heat conduction, the grouping  $k = K/\rho C$  of physical parameters is referred to as the **thermal diffusivity**, and the equation (2.4) is called the **heat equation**. For the remainder of this section we will use the term heat equation to refer to the equation

$$\partial_t u(\vec{x}, t) - \nabla^2 u(\vec{x}, t) = 0.$$

### 3. Some Problems for the Heat Equation

Various side conditions can be adjoined to the heat equation to produce a problem which has one and only one solution for appropriate data. However, not all these problems are well posed.

#### (a) The Cauchy Initial Value problem-

$$\begin{aligned} \partial_t u(\vec{x}, t) - \nabla^2 u(\vec{x}, t) &= f(\vec{x}, t), & \text{for } \vec{x} \in R^n, t > 0, \\ u(\vec{x}, 0) &= g(\vec{x}) & \text{for } \vec{x} \in R^n. \end{aligned}$$

#### (b) An Initial-boundary Value problem-

$$\begin{aligned} \partial_t u(\vec{x}, t) - \nabla^2 u(\vec{x}, t) &= f(\vec{x}, t), & \text{for } \vec{x} \in U, t > 0, \\ u(\vec{x}, 0) &= g(\vec{x}) & \text{for } \vec{x} \in U, \\ B[u(\vec{x}, t)] &= p(\vec{x}, t) & \text{for } \vec{x} \in \partial U, t > 0. \end{aligned}$$

where  $B[u]$  denotes one of the following boundary conditions

$$\text{boundary conditions} \left\{ \begin{array}{l} u(\vec{x}, t) = p(x, t) \\ \partial_{Nu}(\vec{x}, t) = p(x, t) \\ \alpha u(\vec{x}, t) + \beta \partial_{Nu}(\vec{x}, t) = p(x, t) \end{array} \right\}$$

#### (c) The Backward Heat Equation-

$$\begin{aligned}\partial_t u(\vec{x}, t) - \nabla^2 u(\vec{x}, t) &= f(\vec{x}, t), & \text{for } \vec{x} \in R^n, 0 < t < T, \\ u(\vec{x}, T) &= g(\vec{x}) & \text{for } \vec{x} \in R^n.\end{aligned}$$

**(d) The Sideways Heat Equation-**

$$\begin{aligned}\partial_t u(x, t) - \partial_{xx} u(x, t) &= 0, & 0 < x < 1, 0 < t < T, \\ u(0, t) &= p(t) & 0 < t < T, \\ \partial_x u(0, t) &= q(t), & 0 < t < T.\end{aligned}$$

Problems (a) and (b) are well posed while (c) and (d) are examples of problems which are not well posed.

**Problem 14** Consider (c) in the 1-dimensional case with  $f(x, t) = 0$  and  $g(x) = \frac{1}{n} \sin(nx)$ , for  $n$  a positive integer. Show that

$$u_n(x, t) = \frac{1}{n} \exp[n^2(T - t)] \sin(nx)$$

satisfies

$$\begin{aligned}\partial_t u(x, t) - \partial_{xx} u(x, t) &= 0, & \text{for all } x, 0 < t < T, \\ u(x, T) &= \frac{1}{n} \sin(nx) & \text{for all } x.\end{aligned}$$

Show that by choosing  $n$  sufficiently large, the data can be made arbitrarily close to zero while for any  $t$  between  $T$  and zero, the solution can be arbitrarily large. Explain why this implies lack of continuous dependence of the solution on the data.

**Problem 15** Show that for any positive integer,  $n$

$$u_n(x, t) = \frac{1}{n^2} \exp[nx] \cos(nx + 2n^2 t)$$

satisfies

$$\begin{aligned}\partial_t u(x, t) - \partial_{xx} u(x, t) &= 0, & \text{for } 0 < x < 1, 0 < t < T, \\ u(0, t) &= \frac{1}{n^2} \cos(2n^2 t), & \text{for all } 0 < t < T \\ \partial_x u(0, t) &= \frac{1}{n} (\cos(2n^2 t) - \sin(2n^2 t)) & \text{for all } 0 < t < T.\end{aligned}$$

Show that by choosing  $n$  sufficiently large, the data can be made arbitrarily close to zero while the solution can assume arbitrarily large values at points in the domain. Explain why this implies lack of continuous dependence of the solution on the data.

#### 4. Max-min Principles on Bounded Domains

Let  $U$  denote a bounded, open and connected set in  $R^n$  and for  $T > 0$ , given. let

$$Q_T = \{(x, t) : x \in U, t \in (0, T)\}$$

and  $S_T = \{(x, t) : x \in \partial U, t \in (0, T)\} \cup \{(x, t) : x \in \bar{U}, t = 0\}$ .

Then  $Q_T$  is a cylindrical domain and  $S_T$  is referred to as its **parabolic boundary**. We use the notation  $C^{2,1}(Q_T)$  to denote the functions defined and continuous on the open set,  $Q_T$ , together with their  $x$ -derivatives of order  $\leq 2$ , and their  $t$ -derivative of order 1.  $C(\bar{Q}_T)$  is used

to indicate functions which are continuous on the closed set  $\bar{Q}_T$ . Then for  $u = u(x, t) \in C^{2,1}(Q_T) \cap C(\bar{Q}_T)$  let

$$M_0 = \max_{x \in \bar{U}} u(x, 0), \quad M_1 = \max_{x \in \partial U} \max_{0 \leq t < T} u(x, t), \quad M = \max[M_0, M_1]$$

$$m_0 = \min_{x \in \bar{U}} u(x, 0), \quad m_1 = \min_{x \in \partial U} \min_{0 \leq t < T} u(x, t), \quad m = \min[m_0, m_1]$$

Then  $M$  and  $m$  are the max and min of  $u$  over the parabolic boundary. The strong Max-min principle for Laplace's equation is a consequence of the mean value property for harmonic functions. A related line of reasoning applied to solutions of the heat equation leads to a strong Max-min principle for the heat equation. For most purposes, however, the weak form of the max-min principle is sufficient (and it is much easier to develop) so that is the only result we will present here.

**Theorem 4.1** If  $u = u(x, t) \in C^{2,1}(Q_T) \cap C(\bar{Q}_T)$  then

- (a)  $\partial_t u(x, t) - \nabla^2 u(x, t) \leq 0$  in  $Q_T$  implies  $u(x, t) \leq M$  for  $(x, t) \in \bar{Q}_T$
- (b)  $\partial_t u(x, t) - \nabla^2 u(x, t) \geq 0$  in  $Q_T$  implies  $u(x, t) \geq m$  for  $(x, t) \in \bar{Q}_T$
- (c)  $\partial_t u(x, t) - \nabla^2 u(x, t) = 0$  in  $Q_T$  implies  $m \leq u(x, t) \leq M$  for  $(x, t) \in \bar{Q}_T$

Proof of (a)- Suppose first that  $\partial_t u(x, t) - \nabla^2 u(x, t) < 0$  in  $Q_T$ . Then  $u$  can have no relative max inside  $Q_T$  for at such a point,  $(x_0, t_0)$  we would have

$$\partial_t u(x_0, t_0) = 0, \quad \text{and} \quad \nabla^2 u(x_0, t_0) \leq 0 \quad \text{if} \quad 0 < t_0 < T$$

or

$$\partial_t u(x_0, T) \geq 0, \quad \text{and} \quad \nabla^2 u(x_0, T) \leq 0 \quad \text{if} \quad 0 < t_0 = T.$$

In either of these two cases, we would have  $\partial_t u(x_0, t_0) - \nabla^2 u(x_0, t_0) \geq 0$ , in contradiction to the hypothesis. Now suppose that  $\partial_t u(x, t) - \nabla^2 u(x, t) \leq 0$  in  $Q_T$  and introduce the auxiliary function,  $v(x, t) = u(x, t) + \varepsilon|x|^2$ . Then

$$\partial_t v(x, t) - \nabla^2 v(x, t) = \partial_t u(x, t) - \nabla^2 u(x, t) - 2n\varepsilon \leq 2n\varepsilon < 0 \quad \text{in} \quad Q_T,$$

and it follows from the previous argument that  $v$  has no interior max. Then for all  $(x, t)$  in  $\bar{Q}_T$  we have

$$u(x, t) \leq v(x, t) \leq \max_{S_T} v(x, t) \leq \max_{S_T} u(x, t) + \varepsilon R^2 \leq M + \varepsilon R^2$$

where  $R > 0$  is sufficiently large that  $U \subset B_R(0)$ . Since the estimate holds for all  $\varepsilon > 0$ , it follows that  $u(x, t) \leq M$  for  $(x, t) \in \bar{Q}_T$ .

A similar argument can be used to prove (b) and then (c) follows from (a) and (b) together. ■

There is a version of the extended weak Max-min principle that applies to the heat equation.

**Theorem 4.2** Suppose the nonconstant function,  $u = u(x, t) \in C^{2,1}(Q_T) \cap C(\bar{Q}_T)$  satisfies

$$\partial_t u(x, t) - \nabla^2 u(x, t) = 0 \quad \text{in} \quad Q_T.$$

If  $u$  attains an extreme value at  $(x_0, t_0)$  with  $x_0 \in \partial U$  and  $0 < t_0 < T$ , and if  $\partial U$  has a unique tangent plane at  $x_0$ , then  $\nabla u(x_0, t_0) \cdot \vec{n}(x_0) \neq 0$ .

## 5. Applications of Max-min Principles on Bounded Domains

In this section  $U$ ,  $Q_T$ , and  $S_T$  continue to have the same meaning as in section 4.

**Theorem 5.1** Suppose  $F \in C(\bar{Q}_T)$ ,  $f \in C(\bar{U})$ ,  $g \in C(S_T)$ . Then there exists at most one function  $u = u(x,t) \in C^{2,1}(Q_T) \cap C(\bar{Q}_T)$  such that

$$\begin{aligned} \partial_t u(x,t) - \nabla^2 u(x,t) &= F(x,t) \quad \text{in } Q_T, \\ u(x,0) &= f(x), \quad \text{for } x \in U \\ u(x,t) &= g(x,t), \quad \text{for } (x,t) \in S_T \end{aligned} \quad (5.1)$$

**Proof-** Suppose  $u_1, u_2$  both solve the IBVP, (5.1). Then  $w = u_1 - u_2$  satisfies

$$\begin{aligned} \partial_t w(x,t) - \nabla^2 w(x,t) &= 0 \quad \text{in } Q_T, \\ w(x,0) &= 0, \quad \text{for } x \in U \\ w(x,t) &= 0, \quad \text{for } (x,t) \in S_T. \end{aligned}$$

The initial and boundary condition for  $w$  imply that  $\bar{m}=\bar{M}=0$  and then it follows from Theorem 4.1(c) that  $0 \leq w(x,t) \leq 0$  for all  $(x,t) \in \bar{Q}_T$ . Then  $u_1 = u_2$  on  $\bar{Q}_T$ . ■

**Corollary 5.1-** Under the hypotheses of the theorem, if  $u(x,t)$  is a solution of the IBVP (5.1), then

$$\max_{\bar{Q}_T} |u(x,t)| \leq C_0 + K C_F \quad (5.2)$$

where

$$C_F = \max_{Q_T} |F|, \quad C_f = \max_U |f|, \quad C_g = \max_{S_T} |g|, \quad C_0 = \max[C_f, C_g],$$

and  $K$  denotes a constant depending only on  $Q_T$ .

**Proof-** Let  $v(x,t) = u(x,t) + C_F |x|^2/2n$ . Then

$$\begin{aligned} \partial_t v(x,t) - \nabla^2 v(x,t) &= \partial_t u(x,t) - \nabla^2 u(x,t) - C_F = F(x,t) - C_F \leq 0 \quad \text{in } Q_T, \\ v(x,0) &\leq C_0 + C_F R^2/2n \quad \text{on } U \\ v(x,t) &\leq C_0 + C_F R^2/2n \quad \text{on } S_T \end{aligned}$$

where  $R$  is sufficiently large that the bounded set  $U$  is contained in  $B_R(0)$ . Then the max-min principle implies that

$$u(x,t) \leq v(x,t) \leq C_0 + K C_F \quad \text{for } K = R^2/2n.$$

Similarly, letting  $v(x,t) = u(x,t) - C_F |x|^2/2n$  leads to the result,  $u(x,t) \geq v(x,t) \geq -C_0 - K C_F$ . The two results, together, imply  $|u(x,t)| \leq C_0 + K C_F$  for  $(x,t) \in \bar{Q}_T$ . ■

The a-priori estimate (5.2) implies that the solution of (5.1) depends continuously on the data. To see this, define a norm on the data space,  $C(\bar{Q}_T) \times C(\bar{U}) \times C(S_T)$  by

$$\|(F_1, f_1, g_1) - (F_2, f_2, g_2)\|_{data} = \max_{Q_T} |F_1 - F_2| + \max_U |f_1 - f_2| + \max_{S_T} |g_1 - g_2|.$$

and define a norm on the space  $C(\bar{Q}_T)$  by

$$\|u_1 - u_2\|_{C(Q_T)} = \max_{Q_T} |u_1 - u_2|.$$

Since the solution belongs to the subspace  $C^{2,1}(Q_T) \cap C(\bar{Q}_T)$  of the linear space  $C(\bar{Q}_T)$ , we can take  $C(\bar{Q}_T)$  for the solution space. Then the a-priori estimate (5.2) implies that when  $(F_1, f_1, g_1)$  is close to  $(F_2, f_2, g_2)$  in the data space, then the corresponding solutions  $u_1, u_2$  must be close in the solution space; i.e., the mapping which associates the data  $(F, f, g)$  with the solution  $u$  is continuous.

In contrast to this is the problem

$$\begin{aligned} \partial_t u(x, t) - \partial_{xx} u(x, t) &= 0, & 0 < x < \pi, \quad 0 < t < T, \\ u(x, T) &= \frac{1}{n} \sin(nx), & 0 < x < \pi, \\ u(0, t) = u(\pi, t) &= 0, & 0 < t < T, \end{aligned}$$

for which the solution is  $u(x, t) = \frac{1}{n} \sin(nx) \exp[n^2(T - t)]$ . Then the distance from zero data to  $(F, f, g) = (0, \frac{1}{n} \sin(nx), 0)$  is not more than  $\frac{1}{n}$  and so can be made arbitrarily small by choosing  $n$  large. On the other hand,

$$\|u(x, t) - 0\|_{C(Q_T)} \leq \max |\frac{1}{n} \sin(nx) \exp[n^2(T - t)]| = \frac{1}{n} \exp[n^2 T]$$

which shows that as  $n$  grows, distance from zero to the solution grows arbitrarily large. The solution to this problem does not depend continuously on the data.

**Problem 16** Show that there can be at most one admissible solution for the problem

$$\begin{aligned} \partial_t u(x, t) - \nabla^2 u(x, t) &= F(x, t) \quad \text{in } Q_T, \\ u(x, 0) &= f(x), \quad \text{for } x \in U \\ \partial_N u(x, t) &= g(x, t) \quad \text{for } (x, t) \in S_T \end{aligned}$$

What is the class of admissible solutions in this case?

**Problem 17** Suppose that the boundary of  $U$  consists of complementary portions,  $\partial U_1$  and  $\partial U_2$ , and let  $S_{Tj} = \partial U_j \times (0, T)$  for  $j = 1, 2$ . Show that there can be at most one admissible solution for the problem

$$\begin{aligned} \partial_t u(x, t) - \nabla^2 u(x, t) &= F(x, t) \quad \text{in } Q_T \\ u(x, 0) &= f(x), \quad \text{for } x \in U \\ u(x, t) &= g_1(x, t) \quad \text{for } (x, t) \in S_{T,1} \\ \partial_N u(x, t) &= g_2(x, t) \quad \text{for } (x, t) \in S_{T,2} \end{aligned}$$

What is the class of admissible solutions in this case? Does the uniqueness continue to hold if  $\partial U_2 = \partial U$ , i.e., if the boundary conditions are purely Neumann conditions. Recall that for the Laplacian, the solution to the Neumann problem is not unique.

## 6. Max-min Principles on Unbounded Domains

Notice that the function

$$f(t) = \begin{cases} e^{-1/t^2} & \text{for } t > 0 \\ 0 & \text{for } t = 0 \end{cases}$$

satisfies  $f^{(m)}(0) = 0$  for  $m = 0, 1, \dots$  Now let

$$u(x, t) = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} f^{(k)}(t) \quad \text{for all } x, t > 0.$$

Then  $u(x, t) = f(t) + (x^2/2!)f'(t) + (x^4/4!)f''(t) + (x^6/6!)f'''(t) + \dots$

and  $\partial_t u(x, t) = f'(t) + (x^2/2!)f''(t) + (x^4/4!)f^{(3)}(t) + (x^6/6!)f^{(4)}(t) + \dots$

$$\partial_x u(x, t) = xf'(t) + (x^3/3!)f''(t) + (x^5/5!)f^{(3)}(t) + (x^7/7!)f^{(4)}(t) + \dots$$

$$\partial_{xx} u(x, t) = f''(t) + (x^2/2!)f'''(t) + (x^4/4!)f^{(4)}(t) + \dots$$

All of the series above can be shown to converge uniformly on bounded sets in  $R^2$  and it is then evident that  $u(x, t)$  solves the heat equation for all  $x$  and  $t > 0$ . Also,  $u(x, t)$  tends to zero as  $t$  tends to zero so  $u(x, t)$  solves the initial value problem

$$\partial_t u(x, t) - \partial_{xx} u(x, t) = 0 \quad \text{for all } x \in R, t > 0, \quad \text{and} \quad u(x, 0) = 0, \quad \text{for all } x \in R$$

although  $u(x, t)$  is not the zero function. Evidently, there can be no max-min principle on  $R \times R_+$  (nor on  $R^n \times R_+$ ) nor is there uniqueness for the initial value problem for the heat equation if this solution is admissible. It is necessary then to impose some constraint that will disqualify this solution in order to have results for solutions on unbounded sets that is similar to the solution behavior for bounded sets. We will give two different examples of conditions that can be imposed on the solution  $u(x, t)$ , each of which leads to a max-min principle on an unbounded domain. The simpler condition, that the solution is uniformly bounded on the domain, is more restrictive than we would like. The second condition allows the solution to grow without bound so long as it does not grow "too rapidly". For  $T$  positive but finite, consider an unbounded set of the form,  $R_T = \{(x, t) : x \in R^n, 0 \leq t < T\}$ .

We will show that for a function which is bounded on  $\bar{R}_T$  and satisfies the heat equation there, the max and min values of the function on the interior of the set cannot exceed the max and min on  $\{(x, t) : x \in R^n, 0 = t\}$ .

**Theorem 6.1** Suppose  $u \in C^{2,1}(R_T) \cap C(\bar{R}_T)$  and let  $M, m$  denote the maximum and minimum values for  $u(x, t)$  on  $R_T$ . If  $M$  and  $m$  are both finite, then

$$(a) \partial_t u(x, t) - \nabla^2 u(x, t) \leq 0 \quad \text{in } R_T \quad \text{implies} \quad M \leq M_0 = \max_{t=0} u(x, t)$$

$$(b) \partial_t u(x, t) - \nabla^2 u(x, t) \geq 0 \quad \text{in } R_T \quad \text{implies} \quad m \geq m_0 = \min_{t=0} u(x, t)$$

$$(c) \partial_t u(x, t) - \nabla^2 u(x, t) = 0 \quad \text{in } R_T \quad \text{implies} \quad m_0 \leq m \leq M \leq M_0$$

**Proof of (a)-** For  $\varepsilon > 0$ , let  $v(x, t) = u(x, t) - \varepsilon(2nt + |x|^2)$ . Then

$$\partial_t v(x, t) - \nabla^2 v(x, t) = \partial_t u(x, t) - \nabla^2 u(x, t) - \varepsilon(2n - 2n) \leq 0 \quad \text{in } R_T.$$

Also, for all  $x$  in  $R^n$ ,  $v(x, 0) = u(x, 0) - \varepsilon|x|^2 \leq u(x, 0) \leq M_0$ .

Now suppose that  $M > M_0$  and let  $U$  denote the following subset of  $R_T$

$$U = \{(x, t) : |x|^2 \geq (M - M_0)/\varepsilon, 0 \leq t \leq T\}.$$

Then, at each point in  $U$ , we have

$$v(x, t) = u(x, t) - \varepsilon(2nt + |x|^2) \leq M - \varepsilon|x|^2 < M_0$$

while on the complementary set  $R_T \setminus U$ , where  $|x|^2 < (M - M_0)/\varepsilon$ , we have  $v(x, t) \leq v(x, 0) \leq M_0$  by theorem 4.1(a). This shows that  $v(x, t) \leq M_0$  on both the sets,  $U$  and  $R_T \setminus U$ , which implies that  $v(x, t) \leq M_0$  on all of  $R_T$ . Then for all  $\varepsilon > 0$ , and all  $(x, t) \in R_T$ , we have

$$u(x, t) = v(x, t) + \varepsilon(2nt + |x|^2) \leq M_0 + \varepsilon(2nt + |x|^2).$$

Since this holds for all positive  $\varepsilon$ , it follows that  $u(x, t) \leq M_0$  in  $R_T$ . This contradicts the original assumption that  $M > M_0$  and we conclude that  $M \leq M_0$  as asserted in (a). The proof of (b) is similar and the (c) follows from (a) and (b) together. ■

The condition that  $u(x, t)$  is bounded on  $R_T$  is too restrictive for some applications. We can prove the result under the more general condition that

$$\text{for constants } C, a > 0, \quad |u(x, t)| \leq C e^{a|x|^2} \quad \text{for } (x, t) \in R_T \quad (6.2)$$

**Theorem 6.2** Suppose  $u \in C^{2,1}(R_T) \cap C(\bar{R}_T)$  satisfies (6.2). Then the conclusions of theorem 6.1 hold.

Proof of (a)- We can suppose that  $4aT < 1$  here, since if this is not the case, then we just divide the interval  $[0, T]$  into subintervals, each of length,  $T' < 1/4a$  and proceed to prove for  $k = 0, 1, \dots, T/T'$ , that

$$u(x, t) \leq \max_{z \in R} u(z, kT') \leq \max_{z \in R} u(z, 0) \quad \text{for } kT' \leq t \leq (k+1)T'$$

So, assuming then that

$$4aT < 1 \quad (6.3i)$$

there exists an  $\varepsilon > 0$  such that

$$4a(T + \varepsilon) < 1 \quad (6.3ii)$$

Now let

$$K(x, t) = (4\pi t)^{-n/2} e^{-x^2/4t} \quad \text{for all } x, \text{ and } t > 0.$$

Also, for fixed  $y$  and  $\mu > 0$ , let

$$\begin{aligned} V_\mu(x, t) &= u(x, t) - \mu K(i(x-y), T + \varepsilon - t) \\ &= u(x, t) - \mu (4\pi(T + \varepsilon - t))^{-n/2} e^{|x-y|^2/4(T+\varepsilon-t)} \end{aligned}$$

Note that

$$\partial_t K(i(x-y), T + \varepsilon - t) - \nabla_x^2 K(i(x-y), T + \varepsilon - t) = 0 \quad \text{for } 0 < t < T,$$

and

$$\partial_t V_\mu(x, t) - \nabla^2 V_\mu(x, t) = \partial_t u(x, t) - \nabla^2 u(x, t) \leq 0 \quad \text{in } R_T$$

For  $b > 0$ , let  $Q = \{(x, t) \in R_T : |x - y| < b, 0 < t < T\}$ , and use Theorem 4.1(a) on the bounded set,  $Q$ , to conclude that for  $(x, t) \in Q$ ,  $V_\mu(x, t)$  does not exceed the maximum value

of  $V_\mu(x, t)$  over the parabolic boundary of  $Q$ . Note that since  $\mu K(i(x - y), T + \varepsilon - t) > 0$  then on the "bottom" of  $Q$  we have  $V_\mu(x, 0) \leq u(x, 0) \leq M_0$ . On the lateral part of the boundary surface of  $Q$  we have  $|x - y| = b$ ,  $0 < t < T$ , hence it follows from (6.2) and the way we defined  $V_\mu(x, t)$ , that

$$\begin{aligned} V_\mu(x, t) &\leq C e^{a|x|^2} - \mu(4\pi(T + \varepsilon - t))^{-n/2} e^{b^2/4(T + \varepsilon - t)} \\ &\leq C e^{a(|y|+b)^2} - \mu(4\pi(T + \varepsilon - t))^{-n/2} e^{b^2/4(T + \varepsilon - t)} \leq M_0 \end{aligned}$$

for sufficiently large  $b$ . This last inequality follows from the fact that (6.3ii) implies  $b^2/4(T + \varepsilon) > ab^2$ . Then over the complete parabolic boundary of  $Q$  we have  $V_\mu(x, t) \leq M_0$ . By theorem 4.1(a) this estimate holds for all  $(x, t)$  in  $Q$ . In particular, at  $(y, t)$ , then

$$V_\mu(x, t) \leq u(y, t) - \mu(4\pi(T + \varepsilon - t))^{-n/2} \leq M_0$$

i.e.,

$$u(y, t) \leq M_0 + \mu(4\pi(T + \varepsilon - t))^{-n/2} \text{ for any } (y, t) \in R_T$$

Since this last estimate holds for all  $\mu > 0$ , the result (a) follows. ■

We can show that for some  $\alpha > 0$  the function  $u(x, t)$  defined in (6.1) grows in  $R_T$  like  $\exp[x^2/\alpha t]$ , hence it does not satisfy (6.2) for any constant  $a$ . Then the solution to the pure initial-value problem (IVP) for the heat equation is unique in the class of functions which satisfy (6.2). It is also true that the solution to the IVP for the heat equation is unique in the class of functions which are bounded on  $R_T$  but for some initial conditions, the problem may fail to have a bounded solution. It is for this reason we have developed the more general theorem 6.2.

**Problem 18** Show that there can be at most one admissible solution for the problem

$$\begin{aligned} \partial_t u(x, t) - \partial_{xx} u(x, t) &= F(x, t) \quad \text{in } 0 < x < \infty, 0 < t < T \\ u(x, 0) &= f(x), \quad \text{for } x > 0 \\ u(0, t) &= g(t) \quad \text{for } 0 < t < T \end{aligned}$$

What is the class of admissible solutions in this case?

**Problem 19** Can there be more than one admissible solution for the problem

$$\begin{aligned} \partial_t u(x, t) - \partial_{xx} u(x, t) &= F(x, t) \quad \text{in } 0 < x < \infty, 0 < t < T \\ u(x, 0) &= f(x), \quad \text{for } x > 0 \\ \partial_x u(0, t) &= g(t) \quad \text{for } 0 < t < T \end{aligned}$$

What is the class of admissible solutions in this case?

**Problem 20** Suppose  $u_1, u_2$  are solutions for

$$\begin{aligned}\partial_t u(x,t) - \partial_{xx} u(x,t) &= F(x,t) \quad \text{in } 0 < x < L, 0 < t < T \\ u(x,0) &= f(x), \quad \text{for } 0 < x < L \\ u(0,t) &= g(t) \quad \text{for } 0 < t < T \\ \partial_x u(L,t) &= 0 \quad \text{for } 0 < t < T\end{aligned}$$

corresponding to data  $(F_1, f_1, g_1)$  and  $(F_2, f_2, g_2)$  respectively where the data satisfy

$$\begin{aligned}F_1(x,t) &\geq F_2(x,t) \quad \text{in } 0 < x < L, 0 < t < T \\ f_1(x) &\geq f_2(x), \quad \text{for } 0 < x < L \\ g_1(t) &\geq g_2(t) \quad \text{for } 0 < t < T.\end{aligned}$$

Show that  $u_1(x,t) \geq u_2(x,t)$  in  $0 < x < L, 0 < t < T$ .

## 7. Energy Arguments

Suppose  $u \in C^{2,1}(Q_T) \cap C(\bar{Q}_T)$  satisfies

$$\begin{aligned}\partial_t u(x,t) - \nabla^2 u(x,t) &= 0 \quad \text{in } Q_T \\ u(x,0) &= 0, \quad \text{for } x \in U \\ u(x,t) &= 0 \quad \text{for } (x,t) \in S_T\end{aligned}$$

and define

$$E(t) = \int_U u(x,t)^2 dx, \quad 0 \leq t \leq T \quad (7.1)$$

Then the initial condition implies  $E(0) = 0$  and

$$\begin{aligned}E'(t) &= \int_U 2u(x,t) \partial_t u(x,t) dx = \int_U 2u(x,t) \nabla^2 u(x,t) dx, \quad 0 \leq t \leq T. \\ &= 2 \int_{\partial U} u(x,t) \partial_N u(x,t) dS(x) - 2 \int_U \nabla u(x,t) \cdot \nabla u(x,t) dx,\end{aligned}$$

The boundary integral vanishes as a result of the boundary condition and we have then

$$E'(t) = \int_U 2u(x,t) \nabla^2 u(x,t) dx = -2 \int_U |\nabla u(x,t)|^2 dx \leq 0, \quad 0 \leq t \leq T.$$

This implies that  $E(t)$  is nonincreasing and, since  $E(0) = 0$ , it follows that  $E(t)$  is identically zero. Then  $u(x,t)$  is zero in  $\bar{Q}_T$ . The integral in (7.1) is called an **energy integral** and this approach provides an alternative argument for proving uniqueness for the solution of problems for the heat equation on bounded domains. For proving uniqueness for pure initial value problems we must modify the approach by defining

$$E(t) = \int_{R^n} u(x,t)^2 dx, \quad t \geq 0 \quad (7.2)$$

Of course this improper integral fails to converge in general, hence this approach is limited to proving uniqueness for solutions of the IVP in the class of solutions,  $u(x,t)$ , for which the integral in (7.2) is convergent. This may exclude even many bounded solutions.

**Problem 21** Show that there can be at most one admissible solution for the problem

$$\begin{aligned}\partial_t u(x,t) - \partial_{xx} u(x,t) - \partial_x u(x,t) &= F(x,t) \quad \text{in } 0 < x < L, 0 < t < T \\ u(x,0) &= f(x), \quad \text{for } 0 < x < L \\ u(0,t) &= g(t) \quad \text{for } 0 < t < T \\ \partial_x u(L,t) &= h(t) \quad \text{for } 0 < t < T\end{aligned}$$

What is the class of admissible solutions in this case?

**Problem 22** Under what constraints on the constant  $\alpha$  is there at most one admissible solution for the problem

$$\begin{aligned}\partial_t u(x,t) - \partial_{xx} u(x,t) - \partial_x u(x,t) &= F(x,t) \quad \text{in } 0 < x < L, 0 < t < T \\ u(x,0) &= f(x), \quad \text{for } 0 < x < L \\ \partial_x u(L,t) + \alpha u(L,t) &= g(t) \quad \text{for } 0 < t < T\end{aligned}$$

What is the class of admissible solutions in this case?

## 8. A Fundamental Solution for the Heat Equation

Suppose that  $u = u(x,t)$  solves  $\partial_t u(x,t) - \nabla^2 u(x,t) = 0$  in  $R^n \times R_+$ . If  $y = x/\lambda$ ,  $s = t/\lambda^2$  then  $v(y,s) = u(\lambda y, \lambda^2 s)$  satisfies

$$\partial_s v(y,s) = \partial_t u(x,t) \lambda^2, \quad \nabla_y^2 v(y,s) = \nabla_x^2 u(x,t) \lambda^2$$

and

$$\partial_s v(y,s) - \nabla_y^2 v(y,s) = [\partial_t u(x,t) - \nabla_x^2 u(x,t)] \lambda^2 = 0.$$

This suggests that the heat equation may have a solution of the form

$$u(x,t) = t^{-p} \phi(z), \quad \text{where } z = |x|^2/t;$$

i.e.,

$$\partial_t z = -|x|^2/t^2 = -z/t, \quad \text{and} \quad \partial_{x_i} z = 2x_i/t, \quad \partial_{x_i x_i} z = 2/t$$

$$\partial_t u(x,t) = -p t^{-(p+1)} \phi(z) + t^{-p} \phi'(z) \partial_t z = -t^{-(p+1)} (p\phi(z) + z\phi'(z))$$

$$\partial_{x_i} u(x,t) = t^{-p} \phi'(z) \partial_{x_i} z$$

$$\partial_{x_i x_i} u(x,t) = t^{-p} (\phi''(z) (\partial_{x_i} z)^2 + \phi'(z) \partial_{x_i x_i} z)$$

$$\nabla^2 u(x,t) = t^{-(p+1)} (4z\phi''(z) + 2n\phi'(z)),$$

and

$$\begin{aligned}\partial_t u(x,t) - \nabla_x^2 u(x,t) &= -t^{-(p+1)} (p\phi(z) + z\phi'(z) + 4z\phi''(z) + 2n\phi'(z)) \\ &= -t^{-(p+1)} [p(\phi(z) + \frac{2n}{p}\phi'(z)) + z(4\phi''(z) + \phi'(z))]\end{aligned}$$

If  $2n = 4p$ , and  $4\phi''(z) + \phi'(z) = 0$ , then we have  $\partial_t u(x,t) - \nabla_x^2 u(x,t) = 0$ ; i.e., if  $p = n/2$ ,  $\phi(z) = e^{-z/4}$ . Then  $u(x,t)$  is a solution of the heat equation if

$$u(x,t) = C t^{-n/2} e^{-|x|^2/4t} = H_n(x,t),$$

where the constant,  $C$ , is chosen such that  $\int H_n(x,t) dx = 1$  for all  $t > 0$ . That is,

$$\begin{aligned}\int_{R^n} H_n(x,t) dx &= C t^{-n/2} \int_{R^n} e^{-|x|^2/4t} dx = 2^n C \int_{R^n} e^{-|z|^2} dz \\ &= 2^n C \prod_{i=1}^n \int_{R^1} e^{-|z_i|^2} dz_i = 2^n C \pi^{n/2} = 1\end{aligned}$$

hence  $C = (4\pi)^{-n/2}$  and

$$H_n(x,t) = (4\pi t)^{-n/2} e^{-|x|^2/4t} \quad \text{for } x \in R^n, t > 0 \quad (8.1)$$

**Theorem 8.1** Suppose  $g(x)$  is bounded and continuous on  $R^n$ . Then

$$u(x,t) = \int_{R^n} H_n(x-y,t) g(y) dy \quad (8.2)$$

satisfies: i)  $u \in C^\infty(R^n \times R_+)$

ii)  $\partial_t u(x,t) - \nabla^2 u(x,t) = 0$  in  $R^n \times R_+$ .

iii) for all  $x_0 \in R^n$ ,  $u(x,t) \rightarrow g(x_0)$  as  $(x,t) \rightarrow (x_0,0)$

**Proof-** For all  $\delta > 0$ ,  $H_n(x,t)$  is infinitely differentiable with respect to both  $x$  and  $t$  and has uniformly bounded derivatives of all orders in  $R^n \times [\delta, \infty)$ . Since this holds for all  $\delta > 0$ , it then follows that  $u(x,t)$  is infinitely differentiable with respect to both  $x$  and  $t$  for all  $x$  and all positive  $t$ .

For each fixed  $y \in R^n$ ,  $H_n(x-y,t)$  solves the heat equation in the variables  $(x,t)$  hence, for all  $(x,t) \in R^n \times R_+$

$$\partial_t u(x,t) - \nabla^2 u(x,t) = \int_{R^n} (\partial_t - \nabla_x^2) H_n(x-y,t) g(y) dy = 0.$$

To prove iii), fix  $x_0 \in R^n$  and  $\varepsilon > 0$ , and choose  $\delta > 0$  such that  $|g(x_0) - g(y)| \leq \varepsilon \quad \forall y$  such that  $|x_0 - y| < \delta$ . Then for  $|x_0 - x| < \delta/2$ , we have,

$$\begin{aligned}|u(x,t) - g(x_0)| &= \left| \int_{R^n} H_n(x-y,t) g(y) dy - g(x_0) \right| \\ &= \left| \int_{R^n} H_n(x-y,t) (g(y) - g(x_0)) dy \right| \quad (\text{since } \int_{R^n} H_n dy = 1) \\ &\leq \int_{B_\delta(x_0)} H_n(x-y,t) |g(y) - g(x_0)| dy \\ &\quad + \int_{R^n \setminus B_\delta(x_0)} H_n(x-y,t) |g(y) - g(x_0)| dy\end{aligned}$$

Now

$$\int_{B_\delta(x_0)} H_n(x-y,t) |g(y) - g(x_0)| dy \leq \varepsilon \int_{B_\delta(x_0)} H_n(x-y,t) dy \leq \varepsilon.$$

For  $|x_0 - x| < \delta/2$ , and  $|y - x_0| \geq \delta$ , we have

$$|y - x_0| < |y - x| + \delta/2 \leq |y - x| + 1/2|y - x_0|;$$

i.e.,

$$|y - x_0|/2 \leq |y - x|.$$

Then,

$$\begin{aligned}
\int_{R^n \setminus B_\delta(x_0)} H_n(x-y, t) |g(y) - g(x_0)| dy &\leq 2 \max_{R^n \setminus B_\delta(x_0)} |g| \int_{R^n \setminus B_\delta(x_0)} H_n(x-y, t) dy \\
&\leq Ct^{-n/2} \int_{R^n \setminus B_\delta(x_0)} \exp\{-|x-y|^2/4t\} dy \\
&\leq Ct^{-n/2} \int_{R^n \setminus B_\delta(x_0)} \exp\{-|x_0-y|^2/16t\} dy \\
&\leq Ct^{-n/2} \int_\delta^\infty e^{-r^2/16t} r^{n-1} dr \rightarrow 0 \quad \text{as } t \rightarrow 0+.
\end{aligned}$$

This proves that if  $|x_0 - x| < \delta/2$ , and  $t$  is sufficiently small, then  $|u(x, t) - g(x_0)| \leq 2\varepsilon$ . But this implies iii) ■

Formally, the implication of theorem 8.1 is that  $H_n(x-y, t)$  satisfies

$$\begin{aligned}
(\partial_t - \nabla_x^2)H_n(x-y, t) &= 0 \quad \text{for } x \in R^n, t > 0, & (i) \\
H_n(x, 0+) &= \delta(x) \quad \text{for } x \in R^n & (ii)
\end{aligned} \tag{8.3}$$

Later, within the theory of distributions it will be shown that (8.3) is equivalent to

$$\begin{aligned}
(\partial_t - \nabla_x^2)H_n(x, t) &= \delta(x)\delta(t) \quad \text{for } x \in R^n, t > 0, & (i) \\
H_n(x, t) &= 0 \quad \text{for } x \in R^n, t < 0 & (ii)
\end{aligned} \tag{8.4}$$

## 9. Green's Functions for the Heat Equation

The result (8.4) has the following implication for solving an inhomogeneous IBVP. Suppose  $U$  is a bounded open set in  $R^n$  with a smooth boundary  $\partial U$  and that  $u(x, t)$  solves

$$\begin{aligned}
\partial_t u(x, t) - \nabla_x^2 u(x, t) &= F(x, t) \quad x \in U, 0 < t < T, & (9.1) \\
u(x, 0) &= f(x), \quad x \in U, \\
u(x, t) &= g(x, t), \quad x \in \partial U, 0 < t < T.
\end{aligned}$$

Let  $v(x, t) = H_n(x, t) - \phi(x, t)$  where  $\phi(x, t)$  is a smooth function whose properties remain to be specified. Then it follows from (9.1) that

$$\int_0^t \int_U v(x-y, t-s) [\partial_s u(y, s) - \nabla_y^2 u(y, s)] dy ds = \int_0^t \int_U v(x-y, t-s) F(y, s) dy ds \tag{9.2}$$

But

$$\int_0^t \int_U v(x-y, t-s) \partial_s u(y, s) dy ds = \int_U v(x-y, t-s) u(y, s) \Big|_{s=0}^{s=t} dy - \int_0^t \int_U \partial_s v(x-y, t-s) u(y, s) dy ds,$$

and

$$\begin{aligned}
\int_0^t \int_U v(x-y, t-s) \nabla_y^2 u(y, s) dy ds &= \int_0^t \int_U \nabla_y^2 v(x-y, t-s) u(y, s) dy ds \\
&+ \int_0^t \int_{\partial U} v(x-y, t-s) \partial_N u(y, s) dS(y) ds \\
&- \int_0^t \int_{\partial U} \partial_N v(x-y, t-s) u(y, s) dS(y) ds.
\end{aligned}$$

If we now choose  $\phi(x, t)$  so that

$$\begin{aligned} (\partial_t - \nabla_x^2)\phi(x - y, t - s) &= 0 && \text{for } x, y \in U, t > s > 0, \\ \phi(x - y, t - s) &= 0 && \text{for } x, y \in U, t < s, \\ \phi(x - y, t - s) &= H_n(x - y, t - s) && \text{for } x \in U, y \in \partial U, t > s > 0, \end{aligned} \quad (9.3)$$

then

$$\begin{aligned} v(x - y, t - s) &= 0 && \text{for } x \in U, y \in \partial U, t > s > 0, \\ v(x - y, t - s) &= 0 && \text{for } x, y \in U, t = s, \end{aligned}$$

and the equation (9.2) reduces to

$$\begin{aligned} \int_0^t \int_U v(x - y, t - s) F(y, s) dy ds &= - \int_U v(x - y, t) f(y) dy \\ &\quad - \int_0^t \int_U [\partial_s v(x - y, t - s) + \nabla_y^2 v(x - y, t - s)] u(y, s) dy ds \\ &\quad + \int_0^t \int_{\partial U} \partial_N v(x - y, t - s) g(y, s) dS(y) ds. \end{aligned}$$

Finally

$$\partial_s v(x - y, t - s) + \nabla_y^2 v(x - y, t - s) = -(\partial_t - \nabla_x^2)v(x - y, t - s) = -\delta(x - y)\delta(t - s) + 0$$

hence

$$\int_0^t \int_U v(x - y, t - s) F(y, s) dy ds = - \int_U v(x - y, t) f(y) dy + u(x, t) + \int_0^t \int_{\partial U} \partial_N v(x - y, t - s) g(y, s) dS(y) ds$$

The function  $G(x - y, t - s) = H_n(x - y, t - s) - \phi(x - y, t - s)$  for  $\phi$  satisfying (9.3), is referred to as the Green's function for the IBVP (9.1). Formally,  $G$  satisfies

$$\begin{aligned} (\partial_t - \nabla_x^2)G(x - y, t - s) &= \delta(x - y)\delta(t - s) && \text{for } x \in U, 0 < t < T, \\ G(x - y, t - s) &= 0 && \text{for } x, y \in U, s \geq t, \\ G(x - y, t - s) &= 0 && \text{for } x \in U, y \in \partial U, 0 < s < t. \end{aligned}$$

Evidently, if  $u(x, t)$  solves (9.1) then

$$\begin{aligned} u(x, t) &= \int_0^t \int_U G(x - y, t - s) F(y, s) dy ds && (9.4) \\ &\quad + \int_U G(x - y, t) f(y) dy - \int_0^t \int_{\partial U} g(y, s) \partial_N G(x - y, t - s) dS(y). \end{aligned}$$

This is the heat equation analogue of the Green's function representation developed previously for the Laplace equation.

## 10. Comparison of Solutions for the Laplace and Heat Equations

The solution of the Dirichlet problem

$$\nabla^2 u(x) = 0, \quad x \in U, \quad u(x) = g(x) \quad x \in \partial U \quad (10.1)$$

is given by

$$u(x) = - \int_{\partial U} \partial_N G(x, y) g(y) dS(y) \quad (10.2)$$

In the special case that  $U$  is the half space,  $y > 0$ , this becomes

$$u(x,y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{g(s)}{(x-s)^2 + y^2} ds \quad (10.3)$$

The solution,  $u$ , here is harmonic in  $U$ , independent of the smoothness of the boundary data,  $g$ . This implies that  $u$  is infinitely differentiable with respect both  $x$  and  $y$  inside  $U$ , a fact that can also be deduced from (10.3). In any case, the interior smoothness of the solution to the Dirichlet problem is independent of the smoothness on the boundary of the data in the problem. Note that this implies that the convergence  $u(x,y) \rightarrow g(x)$  as  $y \rightarrow 0$ , cannot be uniform in  $x$  if  $g$  is not continuous.

Another property of the solution of the Dirichlet problem (10.1), a consequence of the strong max-min principle, is that if  $g(x)$  is nonnegative on  $\partial U$  and is positive only on a very small subset of  $\partial U$ , then  $u(x)$  is positive on all of  $U$ . This can also be deduced from (10.3) when  $U$  is the half space. This "global influence" property can be interpreted to mean that if the data in (10.1) is altered on even a very small subset of the boundary, then the solution of the BVP is changed at **every point** of  $U$ .

These two properties characterize the so called "organic behavior" of solutions to the Laplace equation. These properties seem somehow natural when it is recalled that Laplace's equation models physical systems which are in a state of equilibrium. When a system in a balanced condition is disturbed on even a small part of its boundary, it must go through a complete readjustment of its interior state in order to achieve a new state of equilibrium. These properties are, in fact, characteristic of solutions of a large class of partial differential equations called **elliptic equations**. It is generally the case that properties that pertain to solutions of Laplace's equation are true for solutions to all elliptic equations.

Now compare the solution of (10.1) with the solution of the Cauchy problem

$$\partial_t u(x,t) - \nabla^2 u(x,t) = 0, \quad x \in R^n, \quad 0 < t < T, \quad u(x,0) = g(x) \quad x \in R^n. \quad (10.4)$$

The solution of (10.4) is given by

$$u(x,t) = \int_{R^n} H_n(x-y,t) g(y) dy = (4\pi t)^{-n/2} \int_{R^n} e^{-|x-y|^2/4t} g(y) dy \quad (10.5)$$

If  $n = 1$  this reduces to

$$u(x,t) = (4\pi t)^{-1/2} \int_{R^n} e^{-|x-y|^2/4t} g(y) dy = \int_{R^n} e^{-z^2} g(x - z\sqrt{4t}) dz \quad (10.6)$$

Here, too, the solution  $u$  can be seen to be infinitely differentiable with respect to both  $x$  and  $t$  for all  $x$  and all positive  $t$ , independent of the smoothness of the initial data  $g$ . Note that as  $t \rightarrow 0$ ,  $u(x,t)$  tends to  $g(x)$  at each  $x$ , hence the convergence cannot be uniform if  $g(x)$  is not continuous. Evidently, the solution becomes infinitely smooth immediately after  $t = 0$ , and we refer to this property as the smoothing action of the solution operator for the heat equation. This smoothing action has an unexpected consequence that is similar to the global influence feature associated with the solutions of Laplace's equation.

Suppose that  $g$  in (10.6) is given by

$$g(x) = \left\{ \begin{array}{ll} 1 & \text{if } |x| < \varepsilon \\ 0 & \text{if } |x| > \varepsilon \end{array} \right\}.$$

Then

$$u(x, t) = \int_{R^n} e^{-z^2} g(x - z\sqrt{4t}) dz = \int_{I_\varepsilon(x, t)} e^{-z^2} dz \quad \text{where } I_\varepsilon(x, t) = \left[ \frac{x - \varepsilon}{\sqrt{4t}}, \frac{x + \varepsilon}{\sqrt{4t}} \right].$$

Evidently, for any  $t > 0$  no matter how small, we have  $u(x, t) > 0$ , for all  $x$ . This means that the influence of the initial state,  $g$ , concentrated in the tiny interval  $[-\varepsilon, \varepsilon]$  is immediately propagated to every point  $x \in R$ . While an infinite speed of propagation is very non physical, it can be rationalized by claiming that to a reasonable degree of approximation, the value of  $u$  is zero outside a bounded set so for practical purposes, the speed of propagation is finite.

The two properties of instantaneous smoothing of the data and infinite speed of propagation are characteristic of solutions of the heat equation and could be referred to as "**diffusionlike evolution**". These properties are, in fact, characteristic of solutions of a large class of partial differential equations called **parabolic equations**. It is generally the case that properties that pertain to solutions of the heat equation are true for solutions to all parabolic equations.