Laplace's Equation

1. Equilibrium Phenomena

Consider a general conservation statement for a region U in R^n containing a material which is being transported through U by a flux field, $\vec{F} = F(\vec{x},t)$. Let $u = u(\vec{x},t)$ denote the scalar concentration field for the material (u equals the concentration at $(\vec{x},t)$). Note that u is a scalar valued function while $F(\vec{x},t)$ is a vector valued function whose value at each $(\vec{x},t)$ is a vector whose direction is the direction of the material flow at $(\vec{x},t)$ and whose magnitude is proportional to the speed of the flow at $(\vec{x},t)$. In addition, suppose there is a scalar source density field denoted by $s(\vec{x},t)$. This value of this scalar at $(\vec{x},t)$ indicates the rate at which material is being created or destroyed at $(\vec{x},t)$. If B denotes an arbitrary ball inside U, then for any time interval (t_1,t_2) conservation of material requires that

$$\int_B u(\vec{x},t_2)\,dx = \int_B u(\vec{x},t_1)\,dx - \int_{t_1}^{t_2}\int_{\partial B} F(\vec{x},t)\bullet\vec{n}(\vec{x})\,dS(x)dt + \int_{t_1}^{t_2}\int_B s(\vec{x},t)\,dxdt$$

Now

$$\int_B u(\vec{x},t_2)\,dx - \int_B u(\vec{x},t_1)\,dx = \int_{t_1}^{t_2}\int_B \partial_t u(\vec{x},t)\,dxdt$$

and

$$\int_{t_1}^{t_2}\int_{\partial B} F(\vec{x},t)\bullet\vec{n}(\vec{x})\,dS(x)dt = \int_{t_1}^{t_2}\int_B div\,F(\vec{x},t)\,dxdt$$

hence

$$\int_{t_1}^{t_2}\int_B [\partial_t u(\vec{x},t) + div\,F(\vec{x},t) - s(\vec{x},t)]\,dxdt = 0 \qquad \textit{for all } B\subset U, \textit{ and all } (t_1,t_2). \qquad (1.1)$$

Since the integrand here is assumed to be continuous, it follows that

$$\partial_t u(\vec{x},t) + div\,F(\vec{x},t) - s(\vec{x},t) = 0, \qquad \textit{for all } \vec{x}\in U, \textit{ and all } t. \qquad (1.2)$$

Equation (1.1) is the integral form of the conservation statement, while (1.2) is the differential form of the same statement. This conservation statement describes a large number of physical processes. We consider now a few special cases,

a) Transport $u = u(\vec{x},t)$
$F(\vec{x},t) = u(x,t)\ \vec{V}$ where $\vec{V} =$ constant,
$s(\vec{x},t) = 0.$

In this case, the equation becomes $\partial_t u(\vec{x},t) + \vec{V}\bullet grad\,u(\vec{x},t) = 0$

b) Steady Diffusion $u = u(\vec{x})$
$F(\vec{x},t) = -K\nabla u(x)$ where $K =$ constant > 0,
$s = s(\vec{x}).$

In this case, the equation becomes

$$-K\,div\,grad\,u(\vec{x}) = s(\vec{x}) \qquad or \qquad -K\,\nabla^2 u(\vec{x}) = s(\vec{x}).$$

This is the equation that governs steady state diffusion of the contaminant through the region U. The equation is called ***Poisson's equation*** if $s(\vec{x}) \neq 0$, and ***Laplace*'*s equation*** when $s(\vec{x}) = 0$. These are the equations we will study in this section.

Another situation which leads to Laplace's equation involves a steady state vector field $\vec{V} = \vec{V}(\vec{x})$ having the property that $div\ \vec{V}(x) = 0$. When $\vec{V}$ denotes the velocity field for an incompressible fluid, the vanishing divergence expresses that $\vec{V}$ conserves mass. When $\vec{V}$ denotes the magnetic force field in a magnetostatic field, the vanishing divergence asserts that there are no magnetic sources. In the case that $\vec{V}$ represents the vector field of electric force, the equation is the statement that U contains no electric charges. In addition to the equation $div\ \vec{V}(x) = 0$, it may happen that $\vec{V}$ satisfies the equation, $curl\ \vec{V}(x) = 0$. This condition asserts that the field $\vec{V}$ is conservative (energy conserving). Moreover, it is a standard result in vector calculus that $curl\ \vec{V}(x) = 0$ implies that $\vec{V} = -grad\ u(\vec{x})$, for some scalar field, $u = u(\vec{x})$. Then the pair of equations,

$$div\ \vec{V}(x) = 0 \qquad and \qquad curl\ \vec{V}(x) = 0,$$

taken together, imply that

$$\nabla^2 u(\vec{x}) = 0 \qquad and \qquad \vec{V} = -grad\ u(\vec{x}).$$

We say that the conservative field $\vec{V}$ is "derivable from the potential, $u = u(\vec{x})$". To say that u is a potential is to say that it satisfies Laplace's equation.

The unifying feature of all of these physical models that lead to Laplace's equation is the fact that they are all in a state of equilibrium. Whatever forces are acting in each model, they have come to a state of equilibrium so that the state of the system remains constant in time. If the balance of the system is disturbed then it will have to go through another transient process until the forces once again all balance each other and the system is in a new equilibrium state.

2. Harmonic Functions

A function $u = u(x)$ is said to be ***harmonic*** in $U \subset R^n$ if:

i) $u \in C^2(U)$; i.e., u, together with all its derivatives of order ≤ 2, is continuous in U

ii) $\nabla^2 u(\vec{x}) = 0$ at each point in U

Note that in Cartesian coordinates,

$$div\nabla u(\vec{x}) = [\partial/\partial x_1, \ldots, \partial/\partial x_n] \bullet \begin{bmatrix} \partial u/\partial x_1 \\ \vdots \\ \partial u/\partial x_n \end{bmatrix} = \partial^2 u/\partial x_1^2 + \ldots + \partial^2 u/\partial x_n^2$$

$$= \partial^\top \partial u(\vec{x}) = \nabla^2 u(\vec{x})$$

It is clear from this that all linear functions are harmonic.

A function depending on x only through the radial variable, $r = \sqrt{x_1^2 + \ldots + x_n^2}$ is said to be a ***radial function***. If u is a radial function then

$$\partial u/\partial x_i = u'(r)\,\partial r/\partial x_i \qquad and \qquad \partial r/\partial x_i = \tfrac{1}{2}(x_1^2 + ... + x_n^2)^{-\frac{1}{2}} 2x_i = x_i/r$$

$$\partial^2 u/\partial x_i^2 = u''(r)(\partial r/\partial x_i)^2 + u'(r)\left[\frac{r - x_i x_i/r}{r^2}\right] = u''(r)(\partial r/\partial x_i)^2 + u'(r)\left[\frac{1}{r} - \frac{x_i^2}{r^3}\right]$$

and

$$\nabla^2 u(\vec{x}) = \sum_{i=1}^n \partial^2 u/\partial x_i^2 = u''(r)\sum_{i=1}^n (x_i/r)^2 + u'(r)\sum_{i=1}^n\left[\frac{1}{r} - \frac{x_i^2}{r^3}\right]$$

$$= u''(r) + u'(r)\left[\frac{n}{r} - \frac{1}{r}\right] = u''(r) + \frac{n-1}{r}u'(r)$$

We see from this computation that the radial function $u = u_n(r)$ is harmonic for various n if:

$$n = 1 \qquad u_1''(r) = 0;\ i.e., \qquad u_1(r) = Ar + B$$

$$n = 2: \qquad u_2''(r) + \tfrac{1}{r}u'(r) = \tfrac{1}{r}\tfrac{d}{dr}[r u_2'(r)] = 0;\ i.e., \qquad u_2(r) = C\ln r$$

$$n > 2: \qquad u_n''(r) + \tfrac{n-1}{r}u_n'(r) = r^{1-n}\tfrac{d}{dr}[r^{n-1}u_n'(r)] = 0; \qquad u_n(r) = C r^{2-n}$$

Note also that since $\nabla^2(\partial u/\partial x_i) = \partial/\partial x_i(\nabla^2 u)$, for any i, it follows that every derivative of a harmonic function is itself, harmonic. Of course this presupposes that the derivative exists but it will be shown that every harmonic function is automatically infinitely differentiable so every derivative exists and is therefore harmonic.

It is interesting to note that if u and u^2 are both harmonic, then u must be constant. To see this, write

$$\nabla^2(u^2) = div(grad\,u^2) = div(2u\nabla u) = 2\nabla u \bullet \nabla u + 2u\nabla^2 u = 2|\nabla u|^2$$

Then $\nabla^2(u^2) = 0$ implies $|\nabla u|^2 = 0$ which is to say, u is constant. Evidently, then, the product of harmonic functions need not be harmonic.

It is easy to see that any linear combination of harmonic functions is harmonic so the harmonic functions form a linear space. It is also easy to see that if $u = u(x)$ is harmonic on R^n then for any $z \in R^n$, the translate, $v(x) = u(x - z)$ is harmonic as is the scaled function, $w = w(\lambda x)$ for all scalars λ. Finally, ∇^2 is invariant under orthogonal transformations. To see this suppose coordinates x *and* y are related by

$$Q\vec{x} = \vec{y} = \begin{bmatrix} Q_{11}x_1 + ... + Q_{1n}x_n \\ \vdots \\ Q_{n1}x_1 + ... + Q_{nn}x_n \end{bmatrix}$$

Then

$$\nabla_x = [\partial/\partial x_1, ..., \partial/\partial x_n]$$

and

$$\partial_{x_i} = (\partial y_1/\partial x_i)\,\partial_{y_1} + ... + (\partial y_n/\partial x_i)\,\partial_{y_n} = Q_{1i}\,\partial_{y_1} + ... + Q_{ni}\,\partial_{y_n}$$

$$= \text{(i-th row of Q)}\bullet\nabla_y$$

i.e., $\nabla_x = Q^\top \nabla_y \qquad and \qquad \nabla_x^\top = (Q^\top \nabla_y)^\top = \nabla_y^\top Q$

Then $\nabla_x^2 = \nabla_x^\top \bullet \nabla_x = \nabla_y^\top Q Q^\top \nabla_y = \nabla_y^\top \nabla_y,$ for $QQ^\top = I$. A transformation Q with this property, $QQ^\top = I,$ is said to be an ***orthogonal*** transformation. Such transformations include rotations and reflections.

Problem 6 Suppose u and v are both harmonic on R^3. Show that, in general, the product of u times v is not harmonic. Give one or more examples of a special case where the product does turn out to be harmonic.

3. Integral Identities

Let U denote a bounded, open, connected set in R^n having a smooth boundary, ∂U. This is sufficient in order for the divergence theorem to be valid on U. That is, if $\vec{F}(\vec{x})$ denotes a smooth vector field over U, (i.e., $F \in C(\bar{U}) \cap C^1(U)$) and if $\vec{n}(x)$ denotes the outward unit normal to ∂U at $x \in \partial U,$ then the divergence theorem asserts that

$$\int_U div\vec{F}\ dx = \int_{\partial U} \vec{F} \bullet \vec{n}\, dS(x) \qquad (3.1)$$

Consider the integral identity (3.1) in the special case that $\vec{F}(x) = \nabla u(x)$ for $u \in C^1(\bar{U}) \cap C^2(U)$. Then

$$div\vec{F}(x) = div\nabla u(x) = \nabla^2 u(x)$$

and $\vec{F} \bullet \vec{n} = \nabla u \bullet \vec{n} = \partial_N u(x)$ (the normal derivative of u)

Then (3.1) becomes

$$\int_U \nabla^2 u(x)\, dx = \int_{\partial U} \partial_N u(x)\, dS(x) \qquad (3.2)$$

The identity (3.2) is known as ***Green's first identity***. If functions u *and* v both belong to $C^1(\bar{U}) \cap C^2(U)$ and if $\vec{F}(x) = v(x)\nabla u(x),$ then

$$div\vec{F}(x) = div\ v(x)\nabla u(x) = v(x)\nabla^2 u(x) + \nabla u \bullet \nabla v$$

and $\vec{F} \bullet \vec{n} = v(x)\nabla u \bullet \vec{n} = v\,\partial_N u(x)$

and, with this choice for $\vec{F},$ (3.1) becomes ***Green's second identity***,

$$\int_U [v(x)\nabla^2 u(x) + \nabla u \bullet \nabla v]\ dx = \int_{\partial U} v(x)\,\partial_N u(x)\, dS(x) \qquad (3.3)$$

Finally, writing (3.3) with u and v reversed, and subtracting the result from (3.3), we obtain ***Green's symmetric identity***,

$$\int_U [v(x)\nabla^2 u(x) - u(x)\nabla^2 v(x)]\ dx = \int_{\partial U} [v(x)\,\partial_N u(x) - u(x)\partial_N v(x)]\ dS(x) \qquad (3.4)$$

Problem 7 Let $u = u(x,y,z)$ be a smooth function on R^3 and let A denote a 3 by 3 matrix whose entries are all smooth functions on R^3 Let $\vec{F} = A\nabla u$. If U denotes a bounded open set in R^3 having smooth boundary ∂U, then find a surface integral over the boundary whose value equals the integral of the divergence of $\vec{F}$ over U. If $v = v(x,y,z)$ is also a smooth function on R^3 then write the integral of $v\, div\vec{F}$ over U as the sum of 2 integrals, one of which is a surface integral over ∂U.

4. The Mean Value Theorem for Harmonic Functions

We begin by introducing some notation:

$B_r(a) = \{x \in R^n : |x-a| < r\}$ the open ball of radius r with center at x=a
$\bar{B}_r(a) = \{x \in R^n : |x-a| \le r\}$ the closed ball of radius r with center at x=a
$S_r(a) = \{x \in R^n : |x-a| = r\}$ the surface of the ball of radius r with center at x=a

Let A_n denote the n-dimensional volume of $B_1(0)$. Then $A_2 = \pi, A_3 = 4\pi/3$, and, in general $A_n = \pi^{n/2}/\Gamma(n/2+1)$. Then the volume of the n-ball of radius r is $r^n A_n$. Also let S_n denote the area of the (n-1)-dimensional surface of $B_1(0)$ in R^n, (i.e, S_n is the area of $\partial B_1(0)$).Then $S_n = nA_n$ and the area of $\partial B_r(0)$ is equal to $nA_n r^{n-1}$. In particular, $S_2 r = 2\pi r,\ S_3 r^2 = 4\pi r^2,\ etc.$

We will also find it convenient to introduce the notation

$$\int_{B_r(a)} f(x)\, d\hat{x} = \frac{1}{A_n r^n}\int_{B_r(a)} f(x)\, dx = average\ value\ of\ f(x)\ over\ B_r(a)$$

and

$$\int_{\partial B_r(a)} f(x)\, d\hat{S}(x) = \frac{1}{S_n r^{n-1}}\int_{\partial B_r(a)} f(x)\, dS(x) = average\ value\ of\ f(x)\ over\ \partial B_r(a)$$

Recall that it follows from Green's first identity that if $u(x)$ is harmonic in U, then for any ball, $B_r(a)$ contained in U, we have

$$\int_{\partial B_r(a)} \partial_N u(x)\, dS(x) = \int_{B_r(a)} \nabla^2 u(x)\, dx = 0.$$

This simple observation is the key to the proof of the following theorem.

Theorem 4.1 (***Mean Value Theorem for Harmonic Functions***)
Suppose $u \in C^2(U)$ and $\nabla^2 u(x) = 0$ for every x in the bounded, open set U in R^n. Then for every $B_r(x) \subset U$,

$$u(x) = \int_{\partial B_r(x)} u(y)\, d\hat{S}(y) = \int_{B_r(x)} u(y)\, d\hat{y} \qquad (4.1)$$

i.e., (4.1) asserts that for every x in U, and $r > 0$, sufficiently small that $B_r(x)$ is contained in U, $u(x)$ is equal to the average value of u over the surface, $\partial B_r(x)$, and $u(x)$ is also equal to the average value of u over the entire ball, $B_r(x)$. A function with the property asserted by (4.1) is said to have the ***mean value property***.

Proof- Fix a point x in U and an $r > 0$ such that $B_r(x)$ is contained in the open set U. Let

$$g(r) = \int_{\partial B_r(x)} u(y)\, d\hat{S}(y) = \int_{\partial B_1(0)} u(x+rz)\, d\hat{S}(z).$$

Here we used the change of variable, $y = x + rz,\ or\ z = (y-x)/r$ so as y ranges over , $\partial B_r(x)$, z ranges over $\partial B_1(0)$. Then

$$g'(r) = \int_{\partial B_1(0)} \nabla u(x+rz) \bullet z\ \ d\hat{S}(z). = \int_{\partial B_r(x)} \nabla u(y) \bullet \frac{y-x}{r}\, d\hat{S}(y)$$

It is evident that as y ranges over $\partial B_r(x),\ |y-x| = r,$ hence $(y-x)/r$ is just the outward unit normal to the surface $\partial B_r(x)$ which means that

$$\nabla u(y) \bullet \frac{y-x}{r} = \partial_N u(y).$$

Then

$$g'(r) = \int_{\partial B_r(x)} \partial_N u(y)\, d\hat{S}(y) = \int_{B_r(x)} \nabla^2 u(y)\, d\hat{y} = 0 \quad \text{(since u is harmonic in U)}$$

Now $g'(r) = 0$ implies that $g(r) =$constant which leads to,

$$g(r) = \lim_{t\to 0}\ g(t) = \lim_{t\to 0} \int_{\partial B_1(0)} u(x+tz)\, d\hat{S}(z) = u(x);$$

i.e., $u(x) = \int_{\partial B_r(x)} u(y)\, d\hat{S}(y)$ for all $r > 0$ such that $B_r(x) \subset U$.

Notice that this result also implies,

$$\int_{B_r(x)} u(y)\, dy = \int_0^r \int_{\partial B_t(x)} u(y)\, dS(y)\, dt = \int_0^r u(x)\, S_n t^{n-1}\, dt = u(x) A_n r^n$$

or,

$$u(x) = \frac{1}{A_n r^n} \int_{B_r(x)} u(y)\, dy = \int_{B_r(x)} u(y)\, d\hat{y}$$

which completes the proof of the theorem.■

The converse of theorem 4.1 is also true.

Theorem 4.2 Suppose U is a bounded open, connected set in R^n and $u \in C^2(U)$ has the mean value property; i.e., for every x in U and for each $r > 0$ such that $B_r(x) \subset U$,

$$u(x) = \int_{\partial B_r(x)} u(y)\, d\hat{S}(y).$$

Then $\nabla^2 u(x) = 0$ in U.

Proof- If it is not the case that $\nabla^2 u(x) = 0$ throughout U, then there is some $B_r(x) \subset U$ such that $\nabla^2 u(x)$ is (say) positive on $B_r(x)$. Then for $g(r)$ as in the proof of theorem 4.1,

$$0 = g'(r) = \int_{\partial B_r(x)} \partial_N u(y)\, d\hat{S}(y) = \frac{r}{n} \int_{B_r(x)} \nabla^2 u(y)\, d\dot{y} > 0$$

This contradiction shows there can be no $B_r(x) \subset U$ on which $\nabla^2 u(x) > 0,$ and hence no point in U where $\nabla^2 u(x)$ is different from zero.■

For $u = u(x,y)$ a smooth function of two variables, we have

$$\partial_{xx} u(x,y) \approx (u(x+h,y) - 2u(x,y) + u(x-h,y))/h^2$$

$$\partial_{yy} u(x,y) \approx (u(x,y+h) - 2u(x,y) + u(x,y-h))/h^2$$

hence $h^2 \nabla^2 u(x,y) \approx -4u(x,y) + u(x+h,y) + u(x-h,y) + u(x,y+h) + u(x,y-h)$

Then the equation, $\nabla^2 u(x,y) = 0$ in U, is approximated by the equation,

$$u(x,y) = (u(x+h,y) + u(x-h,y) + u(x,y+h) + u(x,y-h))/4.$$

The expression on the right side of this equation is recognizable as an approximation for

$$\int_{\partial B_r(x)} u(y)\, d\hat{S}(y).$$

Thus, in the discrete setting, the connection between the property of being harmonic and the mean value property is more immediate.

5. Maximum-minimum Principles

The following theorem, known as the ***strong maximum-minimum principle***, is an immediate consequence of the mean value property.

Theorem 5.1 (***strong maximum – minimum principle***) Suppose U is a bounded open, connected set in R^n and u is harmonic in U and continuous on, $\bar{U}$, the closure of U. Let M and m denote, respectively, the maximum and minimum values of u on ∂U. Then either $u(x)$ is constant on $\bar{U}$ (so then $u(x) = m = M$), or else for every x in U we have $m < u(x) < M$.

Proof Let M denote the maximum value of $u(x)$ on $\bar{U}$ and suppose $u(x_0) = M$. If x_0 is inside U then there exists an $r > 0$ such that $B_r(x_0) \subset U$ and $u(x) \leq u(x_0)$ for all $x \in B_r(x_0)$. Suppose there is some y_0 in $B_r(x_0)$ such that $u(y_0) < u(x_0)$. But this contradicts the mean value property since it implies

$$M = u(x_0) = \int_{B_r(x_0)} u(y)\, d\hat{y} < M.$$

It follows that $u(x) = u(x_0)$ for all x in $B_r(x_0)$. Similarly, for any other point $y_0 \in U$, the assumption that $u(y_0) < u(x_0)$ leads to a contradiction of the mean value property. Then if x_0 is an interior point of U we a force to conclude that $u(x)$ is identically equal to M on U and, by continuity, on the closure, $\bar{U}$. On the other hand, if u is not constant on U, then x_0 must lie on the boundary of U.■

Note that if $u = u(x,y)$ satisfies the discrete Laplace equation,

$$u(x,y) = (u(x+h,y) + u(x-h,y) + u(x,y+h) + u(x,y-h))/4,$$

on a square grid, then u can have neither a max nor a min at an interior point of the grid since at such a point, the left side of the equation could not equal the right side. At an interior maximum, the left side would be greater than all four of the values on the right side, preventing equality. A similar situation would apply at an interior minimum. Unless u is constant on the grid, the only possible location for an extreme value is at a boundary point of the grid.

There is a weaker version of theorem 5.1 that is based on simple calculus arguments.

Theorem 5.2 (***Weak Maximum-minimum principle***) Suppose U is a bounded open, connected set in R^n and $u \in C(\bar{U}) \cap C^2(U)$. Let M and m denote, respectively, the maximum and minimum values of u on ∂U. Then

$$(a) \quad -\nabla^2 u(x) \leq 0 \quad in \quad U \quad implies \quad u(x) \leq M \quad for\ all\ x \in \bar{U}$$

$$(b) \quad -\nabla^2 u(x) \geq 0 \quad in \quad U \quad implies \quad u(x) \geq m \quad for\ all\ x \in \bar{U}$$

$$(c) \quad -\nabla^2 u(x) = 0 \quad in \quad U \quad implies \quad m \leq u(x) \leq M \quad for\ all\ x \in \bar{U}$$

Proof of (a): The argument we plan to use can not be applied directly to $u(x)$. Instead, let $v(x) = u(x) + \varepsilon|x|^2$ for $x \in U$ and note that

$$-\nabla^2 v(x) = -\nabla^2 u(x) - 2n\varepsilon < 0 \quad for\ all\ x\ in \quad U.$$

It follows that $v(x)$ can have no interior maximum, since at such a point, x_0 , we would have

$$\partial v/\partial x_i = 0 \quad and \quad \partial^2 v/\partial x_i^2 \le 0, \quad for \quad 1 \le i \le n, \quad x = x_0.$$

This is in contradiction to the previous inequality since it implies $-\nabla^2 v(x) \ge 0$. This allows us to conclude that $v(x)$ has no interior max and $v(x)$ must therefore assume its maximum value at a point on the boundary of U.

Now U is bounded so for some R sufficiently large, we have $U \subset B_R(0)$ and this implies the following bound on $\max_{x\in U} v(x)$,

$$\max_{x\in U} v(x) \le \max_{x\in\partial U} v(x) \le M + \varepsilon|x|^2 \le M + \varepsilon R^2.$$

Finally, we have, $u(x) \le v(x) \le M + \varepsilon R^2$ for all x in U and all $\varepsilon > 0$. Since this holds for all $\varepsilon > 0$, it follows that $u(x) \le M$ for all x in $\bar{U}$.

Statement (b) can be proved by a similar argument, or, by applying (a) to -u. Then (c) follows from (a) and (b).■

In the special case, $n = 1$, it is easy to see why theorem 5.2 holds. In that case $U = (a,b)$ *and* $\nabla^2 u = u''(x)$ and the figure illustrates (a), (b) and (c).

$$(a) \quad u(x) \le M \qquad (b) \quad u(x) \ge m \qquad (c) \quad m \le u(x) \le M$$

The following figure illustrates why it is necessary to have both of the hypotheses, $u \in C(\bar{U})$, *and* $u \in C^2(U)$.

$u \in C(\bar{U})$, but $u \notin C^2(U)$

$u \notin C(\bar{U})$, but $u \in C^2(U)$

If U is not bounded, then the max-min principle fails in general. For example, if U denotes the unbounded wedge, $\{(x,y) : y > |x|\}$ in R^2 then $u(x,y) = y^2 - x^2$ is harmonic in U, equals zero on the boundary of U, but is not the zero function inside U. An extended version of the max-min principle, due to E Hopf, is frequently useful.

Theorem 5.3 Suppose U is a bounded open, connected set in R^n and $u \in C(\bar{U}) \cap C^2(U)$. Suppose also that $\nabla^2 u(x) = 0$ in U and that u is not constant. Finally, suppose U is such that for each point y on the boundary of U, there is a ball, contained in U with y lying on the boundary of the ball. If $u(y) = M$, then $\partial_N u(y) > 0$ and if $u(y) = m$, then $\partial_N u(y) < 0$.

(i.e., at a point on the boundary of U where $u(x)$ assumes an extreme value, the normal derivative does not vanish).

Problem 8 Let $u(x)$ be harmonic on U and let $v(x) = |\nabla u(x)|^2$. Show that $v(x) \leq \max_{x \in \partial U} v(x)$ for $x \in \bar{U}$.
(Hint: compute $\nabla^2 v$ and show that it is non-negative on U)

6. Consequences of the Mean Value Theorem and M-m Principles

Throughout this section, U is assumed to be a bounded open, connected set in R^n. We list now several consequences of the results of the previous two sections.

It is a standard result in elementary real analysis that if a sequence of continuous functions $\{u_m\}$ converges uniformly to a limit u on a compact set K, then u is also continuous. Moreover, for any open subset W in K, we have

$$\lim_{m\to\infty} \int_W u_m\, dx = \int_W u\, dx.$$

Lemma 6.1 Suppose $\{u_m(x)\}$ is a sequence of functions which are harmonic in U and which converge uniformly on $\bar{U}$. Then $u = \lim_{m\to\infty} u_m$ is harmonic in U.

Proof Since each u_m is harmonic in U, theorem 4.1 implies that for every ball, $B_r(x) \subset U$, we have

$$u_m(x) = \int_{\partial B_r(x)} u_m(y)\, d\hat{S}(y) = \int_{B_r(x)} u_m(y)\, d\hat{y}.$$

The uniform convergence of the sequence on U implies that

$$u_m(x) \to u(x), \qquad \int_{\partial B_r(x)} u_m(y)\, d\hat{S}(y) \to \int_{\partial B_r(x)} u(y)\, d\hat{S}(y), \qquad \int_{B_r(x)} u_m(y)\, d\hat{y} \to \int_{B_r(x)} u(y)\, d\hat{y}$$

hence

$$u(x) = \int_{\partial B_r(x)} u(y)\, d\hat{S}(y) = \int_{B_r(x)} u(y)\, d\hat{y}.$$

But this says u has the mean value property and so, by theorem 4.2, u is harmonic.■

Lemma 6.2 Suppose $u \in C(\bar{U}) \cap C^2(U)$ satisfies the conditions

$$\nabla^2 u(x) = 0, \quad in\ U, \quad and \quad u(x) = 0, \quad on\ \partial U.$$

Then $u(x) = 0$ for all x in U.

Proof- The hypotheses, $u \in C(\bar{U}) \cap C^2(U)$ and $\nabla^2 u(x) = 0,\ in\ U,$ imply that $m \leq u(x) \leq M$, in $\bar{U}$. Then $u(x) = 0,\ on\ \partial U$ implies $m = M = 0$.■

Lemma 6.2 asserts that the so called ***Dirichlet boundary value problem***

$$\nabla^2 u(x) = F(x), \quad x \in U, \quad and \quad u(x) = g(x), \quad x \in \partial U,$$

has at most one solution in the class $C(\bar{U}) \cap C^2(U)$. Solutions having this degree of smoothness are called classical solutions of the Dirichlet boundary value problem. The partial differential equation is satisfied at each point of U and the boundary condition is satisfied at each point of the boundary. Later we are going to consider solutions in a wider sense.

Lemma 6.3 For any $F \in C(U)$ *and* $g \in C(\partial U)$, there exists at most one $u \in C(\bar{U}) \cap C^2(U)$ satisfying

$$-\nabla^2 u(x) = F, \quad in\ U, \quad and \quad u(x) = g, \quad on\ \partial U.$$

Proof Suppose $u_1,\ u_2 \in C(\bar{U}) \cap C^2(U)$ both satisfy the conditions of the boundary value problem. Then $w = u_1 - u_2$ satisfies the hypotheses of lemma 6.2 and is therefore zero on the closure of U. Then $u_1 = u_2$ on the closure of U.■

Lemma 6.4 Suppose $u \in C(\bar{U}) \cap C^2(U)$ satisfies

$$\nabla^2 u(x) = 0, \quad in\ U, \quad and \quad u(x) = g, \quad on\ \partial U,$$

where $g(x) \geq 0$. If $g(x_0) > 0$ at some point $x_0 \in \partial U$ then $u(x) > 0$ at every $x \in U$.

Proof First, $g(x) \geq 0$ implies that $m = 0$. Then $g(x_0) > 0$ at some point $x_0 \in \partial U$ implies $M > 0$. It follows now from the strong M-m principle that $0 < u(x) < M$ at every $x \in U$.■

Note that lemma 6.4 asserts that if a harmonic function that is non-negative on the boundary of its domain is positive at ***some*** point of the boundary, then it must be positive at ***every*** point inside the domain; i.e., a local stimulus applied to the ”skin” of the body produces a global response felt everywhere inside the body. This could be referred to as the ***organic behavior*** of harmonic functions. This mathematical behavior is related to the fact that Laplace’s equation models physical systems that are in a state of equilibrium. If the boundary state of a system in equilibrium is disturbed, even if the disturbance is very local, then the system must readjust itself at each point inside the boundary to achieve a new state of equilibrium. This is the physical interpretation of ”organic behavior”.

Lemma 6.5 For $F \in C(\bar{U})$ *and* $g \in C(\partial U)$, suppose $u \in C(\bar{U}) \cap C^2(U)$ satisfies

$$-\nabla^2 u(x) = F(x), \quad x \in U, \quad and \quad u(x) = g(x), \quad x \in \partial U.$$

Then $\max_{x \in U} |u(x)| \leq C_g + MC_F$

where $C_g = \max_{x \in \partial U} |g(x)|, \quad C_F = \max_{x \in U} |F(x)|, \quad M =$ a constant depending on U.

Proof The estimate asserts that $-(C_g + MC_F) \leq u(x) \leq C_g + MC_F$ for $x \in \bar{U}$. First, let

$$v(x) = u(x) + |x|^2 \frac{C_F}{2n}$$

Then $-\nabla^2 v(x) = -\nabla^2 u(x) - C_F = F(x) - C_F \leq 0 \quad in\ U$

and $v(x) \leq \max_{x \in \partial U} (u(x) + |x|^2 \frac{C_F}{2n}) \quad for \quad x \in \bar{U}.$

Since U is bounded, there exists some $R > 0$ such that $|x|^2 \leq R^2$ for $x \in U$. Then

$$v(x) \le C_g + R^2 \frac{C_F}{2n} \qquad and \qquad u(x) \le v(x) \le C_g + MC_F \qquad for x \in \bar{U}.$$

Similarly, let

$$w(x) = u(x) - |x|^2 \frac{C_F}{2n}$$

and show that $u(x) \ge w(x) \ge -(C_g + MC_F) \quad for\ x \in \bar{U}$.■

If we define a mapping, $S : C(\bar{U}) \times C(\partial U) \to C(\bar{U}) \cap C^2(U)$ that associates the data pair, (F,g), for the boundary value problem of lemma 6.5 to the solution $u(x)$, then we would write $u = S[F,g]$. Evidently, lemma 6.5 asserts that the mapping S is continuous. To make this statement precise, we must explain how to measure distance between data pairs $(F_1,g_1),(F_2,g_2)$ in the data space $C(\bar{U}) \times C(\partial U)$ and between solutions u_1,u_2 in the solution space $C(\bar{U})$. Although we know that the solutions belong to the space $C(\bar{U}) \cap C^2(U)$, this is a subspace of the larger space, $C(\bar{U})$, so we are entitled to view the solutions as belonging to this larger space. We are using the term ”space” to mean a linear space of functions; that is, a set that is closed under the operation of forming linear combinations.

Define the distance between u_1,u_2 in the solution space $C(\bar{U})$ as follows

$$\|u_1 - u_2\|_{C(\bar{U})} = \max_{x\in\bar{U}} |u_1(x) - u_2(x)|.$$

Similarly, define the distance from (F_1,g_1) to (F_2,g_2) in the data space $C(\bar{U}) \times C(\partial U)$ by

$$\|(F_1,g_1) - (F_2,g_2)\|_{C(\bar{U})\times C(\partial U)} = \max_{x\in\bar{U}} |F_1(x) - F_2(x)| + \max_{x\in\partial U} |g_1(x) - g_2(x)|..$$

Each of these ”distance functions” defines what is called a ***norm*** on the linear space where it has been defined. In order to be called a norm, the functions have to satisfy the following conditions,

$i)$ $\|\alpha u\| = |\alpha|\|u\|$ for all scalars α and for all functions u

$ii)$ $\|u + v\| \le \|u\| + \|v\|$, for all functions u,v

$iii)$ $\|u\| \ge 0$, for all u and $\|u\| = 0$ if and only if $u = 0$.

One can check that the distance functions defined above both satisfy all three of these conditions and they therefore qualify as norms on the spaces where they have been defined. Now the estimate of lemma 6.5 asserts that if u_j solves the boundary value problem with data (F_j,g_j), $j = 1,2$ then

$$\max_{x\in U} |u_1(x) - u_2(x)| \le \max_{x\in\partial U} |g_1(x) - g_2(x)| + M \max_{x\in U} |F_1(x) - F_2(x)|$$

i.e., $$\|u_1 - u_2\|_{C(\bar{U})} \le \max(1,M)\ \|(F_1,g_1) - (F_2,g_2)\|_{C(\bar{U})\times C(\partial U)}.$$

Evidently, if the data pairs are close in the data space, then the solutions are correspondingly close in the solution space. This is what is meant by continuous dependence of the solution on the data. Note that if we were to change the definition of the norm in one or the other (or both) of the spaces, the solution might no longer depend continuously on the data.

Consider the solution for the following boundary value problem

$$\nabla^2 u(x,y) = 0 \qquad\qquad for \quad 0 < x < \pi,\ y > 0,$$

$$u(x,0) = 0, \qquad \partial_y u(x,0) = g(x) = \tfrac{1}{n}\sin nx, \qquad 0 < x < \pi,$$
$$u(0,y) = u(\pi,y) = 0, \qquad y > 0.$$

For any integer, n, the solution is given by $u(x,y) = \frac{1}{n^2}\sin nx \ \sinh ny.$

Evidently, the distance between g and zero in the data space is

$$\|g(x) - 0\|_{C(R)} = \max_{x\in R}\left|\tfrac{1}{n}\sin nx\right| \le \tfrac{1}{n},$$

while the distance between $u(x,y)$ and zero in the solution space is

$$\|u(x,y) - 0\|_{C(0<x<\pi,\ y>0,)} = \max_{0<x<\pi,\ y>0,}\left|\tfrac{1}{n^2}\sin nx \ \sinh ny\right| \approx \tfrac{e^{ny}}{n^2}$$

This means that the data can be made arbitrarily close to zero by choosing n large, while the solution can simultaneously be made as far from zero as we like by choosing $y > 0$, large. Then the solution to this problem does not depend continuously on the data since arbitrarily small data errors could lead to arbitrarily large solution errors. This problem is said to be ”not well posed”.

7. Uniqueness from Integral Identities

Integral identities can be used to prove that various boundary value problems cannot have more than one solution. For example, consider the following boundary value problem

$$\nabla^2 u(x) = F(x), \qquad x \in U, \qquad \partial_N u(x) = g(x), \qquad x \in \partial U.$$

This is known as the ***Neumann boundary value problem*** for Poisson’s equation. Green’s first identity leads to

$$\int_U F(x)\,dx = \int_U \nabla^2 u(x)\,dx = \int_{\partial U} \partial_N u(x)\,dS(x) = \int_{\partial U} g(x)\,dS(x).$$

Then a necessary condition for the existence of a solution to this problem is that the data, (F,g) satisfies

$$\int_U F(x)\,dx = \int_{\partial U} g(x)\,dS(x).$$

If this condition is satisfied, and if u_1, u_2 denote two solutions to the problem, then $w = u_1 - u_2$ satisfies the problem with $F = g = 0$. Then we have

$$0 = \int_U w\nabla^2 w\,dx = \int_{\partial U} w\,\partial_N w\,dS(x) - \int_U \nabla w \bullet \nabla w\,dx = -\int_U |\nabla w|^2\,dx$$

But this implies that $|\nabla w| = 0$ which is to say, w is constant in U. Then the solutions to this boundary value problem may differ by a constant, they are not unique. We should point out that in order for the equation and the boundary condition to have meaning in the classical sense, we must assume that the solutions to this problem belong to the class, $C^1(\bar{U}) \cap C^2(U)$.

On the other hand, consider the problem,

$$\nabla^2 u(x) = F(x), \qquad x \in U, \qquad u(x) = g_1(x), \quad x \in \partial U_1, \qquad \partial_N u(x) = g_2(x), \qquad x \in \partial U_2,$$

where ∂U is composed of two distinct pieces, ∂U_1, and ∂U_2. Now if u_1, u_2 denote two solutions to the problem, and $w = u_1 - u_2$, then we have, as before

$$0 = \int_U w\nabla^2 w\,dx = \int_{\partial U} w\,\partial_N w\,dS(x) + \int_U \nabla w \bullet \nabla w\,dx = \int_{\partial U_1} w\,\partial_N w\,dS(x) + \int_{\partial U_2} w\,\partial_N w\,dS(x) + \int_U |\nabla w|^2$$

In this case, $w = 0$ on ∂U_1 and $\partial_N w = 0$ on ∂U_2, so we again reach the conclusion that w is constant in U. Since $w \in C^1(\bar{U}) \cap C^2(U)$, it follows that if $w = 0$ *on* ∂U_1, then $w = 0$ on $\bar{U}$. Then the solution to this problem is unique.

Finally, consider the Neumann problem for the so called ***Helmholtz equation***,

$$-\nabla^2 u(x) + c(x)\,u(x) = F(x), \qquad x \in U, \qquad u(x) = g(x), \quad x \in \partial U,$$

where we suppose that $c(x) \geq C_0 > 0$ for $x \in U$. We can use integral identities to show that this problem has at most one smooth solution. As usual, we begin by supposing the problem has two solutions and we let $w(x)$ denote their difference. Then

$$-\nabla^2 w(x) + c(x)\,w(x) = 0, \qquad x \in U, \qquad w(x) = 0, \quad x \in \partial U,$$

and,

$$0 = \int_U w(x)[-\nabla^2 w(x) + c(x)\,w(x)]\,dx = -\int_{\partial U} w\,\partial_N w\,dS(x) + \int_U \nabla w \bullet \nabla w\,dx + \int_U c(x)\,w(x)^2 dx$$

Since $w = 0$ on ∂U, it follows that

$$\int_U (|\nabla w|^2 + c(x)\,w(x)^2)\,dx \geq C_0 \int_U w(x)^2 dx = 0,$$

and this implies that $w(x)$ vanishes at every point of $\bar{U}$. Notice that this proof of uniqueness doesn't work if we don't know that the coefficient $c(x)$ is non-negative. (How would the proof have to be modified if we knew only that $c(x) \geq 0$?).

Problem 9 Prove that the following problem has at most one smooth solution

$$-\nabla^2 u(x) = F(x), \qquad x \in U, \qquad and \qquad u(x) = g(x), \quad x \in \partial U.$$

Use first the Green's identity approach and then use the result in lemma 6.5. Note that this result was already established by means of the M-m principle.

Problem 10 Prove that the following problem has at most one smooth solution

$$-\nabla^2 u(x) = F(x), \qquad in\ U, \qquad and \qquad u(x) + \partial_N u(x) = g(x), \quad on \quad \partial U.$$

Eigenvalues for the Laplacian

The ***eigenvalues*** for the Dirichlet problem for the Laplace operator are any scalars, λ, for which there exist nontrivial solutions to the Dirichlet boundary value problem,

$$-\nabla^2 u(x) = \lambda\,u(x), \quad x \in U, \qquad u(x) = 0, \quad x \in \partial U.$$

Note that if $u(x) = 0$ then any choice of λ will satisfy the conditions of the problem. Therefore we allow only nontrivial solutions and we refer to these as ***eigenfunctions***. If $u(x)$ is an eigenfunction for this problem corresponding to an eigenvalue λ then

$$\lambda \int_U u(x)^2 dx = -\int_U u(x)\nabla^2 u(x)\,dx = -\int_{\partial U} u\partial_N u\,dS(x) + \int_U |\nabla u|^2 dx.$$

Then λ satisfies

$$\lambda = \frac{\int_U |\nabla u|^2 dx.}{\int_U u(x)^2 dx} > 0.$$

Note that $|\nabla u| \neq 0$ since this would lead to $u = 0$ which is not allowed if u is an eigenfunction. We have shown that all eigenvalues of the Dirichlet problem for the Laplace operator are strictly positive.

Problem 11 Show that the Neumann problem,

$$-\nabla^2 u(x) = \lambda u(x), \quad x \in U, \qquad \partial_N u(x) = 0, \quad x \in \partial U.$$

has a zero eigenvalue which has the corresponding eigenfunction, $u(x) =$constant.

Problem 12 Under what conditions on the function $\alpha(x)$, does the boundary value problem,

$$-\nabla^2 u(x) = \lambda u(x), \quad x \in U, \qquad \alpha(x)\, u(x) + \partial_N u(x) = 0, \quad x \in \partial U.$$

have only positive eigenvalues?

Problem 13 Show that for each of the eigenvalue problems considered here, if $u(x)$ is an eigenfunction corresponding to an eigenvalue, λ, then for any nonzero constant k, $v(x) = ku(x)$, is also an eigenfunction corresponding to the eigenvalue, λ.

8. Fundamental Solutions for the Laplacian

Let $\delta(x)$ denote the "function" with the property that for any continuous function, $f(x)$,

$$\int_{R^n} \delta(x) f(x)\, dx = f(0), \quad \text{or, equivalently,} \quad \int_{R^n} \delta(x-y) f(y)\, dy = f(x)$$

Of course this is a purely formal definition since there is no function $\delta(x)$ which could have this property. Later, we will see that $\delta(x)$ can be given a rigorous, consistent meaning in the context of generalized functions. However, using the delta in this formal way, we can give a formal definition of a fundamental solution for the negative Laplacian as the solution of,

$$-\nabla_x^2 E(x-y) = \delta(x-y), \qquad x,y \in R^n. \tag{8.1}$$

Formally, this definition implies

$$-\nabla_x^2 \int_{R^n} E(x-y) f(y)\, dy = \int_{R^n} \delta(x-y) f(y)\, dy = f(x)$$

Then the solution of the equation

$$-\nabla^2 u(x) = f(x), \qquad x \in R^n,$$

is given by

$$u(x) = \int_{R^n} E(x-y) f(y)\, dy. \tag{8.2}$$

Although these steps are only formal, they can be made rigorous. Note that since there are no side conditions imposed on $E(x)$ or on $u(x)$ neither of these functions is unique. For example, any harmonic function could be added to either of them and the resulting function would still satisfy the same equation.

Since $\delta(x)$ and ∇^2 are both radially symmetric, it seems reasonable to assume that $E(x)$ is radially symmetric as well; i.e., $E(x) = E(r)$, for $r = \sqrt{x_1^2 + ... + x_n^2}$. Then a definition for E(x) which does not make use of $\delta(x)$ can be stated as follows:

$E_n(x)$ is a ***fundamental solution for*** $-\nabla^2$ ***on*** R^n if,

$$i) \quad E_n(r) \in C^2(R^n \backslash \{0\})$$

$$ii) \quad \nabla^2 E_n(r) = 0, \qquad for\ r > 0 \qquad (8.3)$$

$$iii) \quad \lim_{\varepsilon \to 0} \int_{\partial B_\varepsilon(0)} \partial_N E_n(x) dS(x) = -1$$

The properties i) and ii) in the definition imply that

$$\nabla^2 E_n(r) = E_n"(r) + \frac{n-1}{r} E_n'(r) = 0, \qquad for\ r > 0$$

i.e., $E_n"(r)/E_n'(r) = -(n-1)/r$

$$\log\left(E_n'(r)\right) = -(n-1)\log r + C,$$

$$E_n'(r) = C r^{1-n},$$

$$E_n(r) = \left\{ \begin{array}{ll} C_2 \log r & if\ n = 2 \\ C_n r^{2-n} & if\ n > 2 \end{array} \right\}.$$

The constant C_n can be determined from part iii) of the definition. It is this part of the definition that causes $-\nabla^2 E_n(x)$ to behave like $\delta(x)$.
For $n = 2$ we have

$$\int_{\partial B_\varepsilon(0)} \partial_N E_n(x) dS(x) = \int_0^{2\pi} \partial_r (C_2 \log r)\, \varepsilon\, d\theta = C_2 \int_0^{2\pi} \tfrac{1}{\varepsilon}\, \varepsilon\, d\theta = 2\pi C_2.$$

Then $\lim_{\varepsilon \to 0} \int_{\partial B_\varepsilon(0)} \partial_N E_2(x) dS(x) = 2\pi C_2. = -1$

so $C_2 = -1//2\pi$ and $E_2(r) = -\frac{1}{2\pi} \log r$.

When $n = 3$ we have

$$\int_{\partial B_\varepsilon(0)} \partial_N E_n(x) dS(x) = \int_{\partial B_\varepsilon(0)} \partial_r (C_3/r)\, \varepsilon^2\, d\omega = -C_3 \int_{\partial B_\varepsilon(0)} \tfrac{1}{\varepsilon^2}\, \varepsilon^2\, d\omega = -4\pi C_3.$$

Then $\lim_{\varepsilon \to 0} \int_{\partial B_\varepsilon(0)} \partial_N E_3(x) dS(x) = -4\pi C_3. = -1$

so $C_3 = 1//4\pi$ and $E_3(r) = 1/4\pi r$.

We will now show that condition (8.3) iii) really does produce the δ behavior for $-\nabla^2 E_n$. Of course we can't try to show that $-\nabla^2 E_n = \delta(x)$ since are not allowed to refer to $\delta(x)$. Instead, we will show equivalently that $-\nabla^2 u(x) = f(x)$, for u given by (8.2). Here, we suppose that $f(x)$ is continuous, together with all its derivatives of order less than or equal to 2, and we suppose further that $f(x)$ has compact support; i.e., for some positive K, $f(x)$ vanishes for $|x| > K$. The notation for this class of functions is $C_c^2(R^n)$.

Theorem 8.1 Let $E_n(r)$ denote a fundamental solution for $-\nabla^2$ *on* R^n. Then, for any $f \in C_c^2(R^n)$,

$$u(x) = \int_{R^n} E_n(x-y) f(y)\, dy,$$

satisfies

$$u \in C^2(R^n), \qquad -\nabla^2 u(x) = f(x) \qquad \text{for any } x \in R^n.$$

Proof The smoothness of f implies the smoothness of u; i.e., for $i = 1, 2, ..., n$

$$\partial u/\partial x_i = \lim_{h\to 0} \frac{u(\vec{x} + h\vec{e}_i) - u(\vec{x})}{h} = \lim_{h\to 0} \int_{R^n} E(z) \frac{f(\vec{x} + h\vec{e}_i - z) - f(\vec{x} - z)}{h}\, dz,$$

Now $\dfrac{f(\vec{x} + h\vec{e}_i - z) - f(\vec{x} - z)}{h}$ converges uniformly to $\partial f/\partial x_i$ and it follows that for each i,

$$\partial u/\partial x_i = \int_{R^n} E(z)\, \partial_{x_i} f(x-z)\, dy,$$

Similarly, $\partial^2 u(x)/\partial x_i \partial x_j$ exists for each i and j since the corresponding derivatives of f all exist.

To show the second assertion, write

$$-\nabla_x^2 u(x) = \int_{R^n} -E_n(z) \nabla_x^2 f(x-z)\, dy = \int_{R^n} -E_n(z)\, \nabla_z^2 f(x-z)\, dz.$$

Since $E_n(z)$ tends to infinity as $|z|$ tends to zero, we treat this as an improper integral;

$$\int_{R^n} E_n(z)\, \nabla_z^2 f(x-z)\, dz = \int_{B_\varepsilon(0)} E_n(z)\, \nabla_z^2 f(x-z)\, dz + \int_{R^n \setminus B_\varepsilon(0)} E_n(z)\, \nabla_z^2 f(x-z)\, dz.$$

First, note that

$$\left| \int_{B_\varepsilon(0)} E_n(z)\, \nabla_z^2 f(x-z)\, dz \right| \le \max_{B_\varepsilon(0)} |\nabla_z^2 f(x-z)| \int_{B_\varepsilon(0)} |E_n(z)|\, dz.$$

But

$$\int_{B_\varepsilon(0)} |E_n(z)|\, dz. = \left\{ \begin{array}{ll} 1/2\pi \int_0^\varepsilon \int_0^{2\pi} |\log r| r dr\, d\theta = C\varepsilon^2 |\log \varepsilon| & \text{if } \; n = 2 \\ C_n \int_0^\varepsilon \int_\omega r^{2-n} r^{n-1} dr\, d\omega = C\varepsilon^2 & \text{if } \; n > 2 \end{array} \right\}$$

hence

$$\lim_{\varepsilon\to 0} \left| \int_{B_\varepsilon(0)} E_n(z)\, \nabla_z^2 f(x-z)\, dz \right| = 0.$$

Next,

$$\int_{R^n \setminus B_\varepsilon(0)} E_n(z)\, \nabla_z^2 f(x-z)\, dz = \int_{\partial(R^n \setminus B_\varepsilon(0))} E_n(z)\, \partial_N f(x-z)\, dS(z) - \int_{R^n \setminus B_\varepsilon(0)} \nabla E_n(z) \bullet \nabla_z f(x-z)\, dz,$$

and

$$\left| \int_{\partial(R^n \setminus B_\varepsilon(0))} E_n(z)\, \partial_N f(x-z)\, dS(z) \right| \le \max_{z \in \partial B_\varepsilon(0)} |\partial_N f(x-z)| \int_{-\partial B_\varepsilon 0)} |E_n(z)|\, dS(z)$$

$$\leq C_1 \begin{cases} 1/2\pi \int_{2\pi}^{0} |\log \varepsilon| \varepsilon \, d\theta = C_2 |\log \varepsilon| \varepsilon & if \quad n = 2 \\ C_n \int \varepsilon^{2-n} \varepsilon^{n-1} d\omega = C_3 \varepsilon & if \quad n > 2 \end{cases}$$

We used the fact that $\partial(R^n \backslash B_\varepsilon(0)) = -\partial B_\varepsilon(0)$. Finally, since $E_n(z)$ is harmonic in $R^n \backslash B_\varepsilon(0)$,

$$\int_{R^n \backslash B_\varepsilon(0)} \nabla E_n(z) \bullet \nabla_z f(x-z)\, dz = \int_{-\partial B_\varepsilon(0)} \partial_N E_n(z) f(x-z)\, dS(z) - \int_{R^n \backslash B_\varepsilon(0)} \nabla^2 E_n(z)\, f(x-z)\, dz$$

$$= \int_{-\partial B_\varepsilon 0)} \partial_N E_n(z) f(x-z)\, dS(z).$$

Now we can write

$$\int_{-\partial B_\varepsilon 0)} \partial_N E_n(z) f(x-z)\, dS(z) = \int_{-\partial B_\varepsilon 0)} \partial_N E_n(z) [f(x-z) - f(x)]\, dS(z) + \int_{-\partial B_\varepsilon 0)} \partial_N E_n(z) f(x)\, dS(z),$$

and note that because $f(x)$ is continuous,

$$\left| \int_{-\partial B_\varepsilon 0)} \partial_N E_n(z) [f(x-z) - f(x)]\, dS(z) \right| \leq C \max_{z \in \partial B_\varepsilon 0)} |f(x-z) - f(x)| \to 0 \quad as \ \varepsilon \to 0.$$

In addition,

$$\int_{-\partial B_\varepsilon 0)} \partial_N E_n(z)\, dS(z) = -\int_{\partial B_\varepsilon 0)} \partial_N E_n(z)\, dS(z) \to 1 \ as \ \varepsilon \to 0$$

because of $(8.3)iii$, and then it follows that

$$-\nabla^2 u(x) = \lim_{\varepsilon \to 0} \int_{-\partial B_\varepsilon 0)} \partial_N E_n(z) f(x-z)\, dS(z) = f(x) \ \ \forall x \in R^n \blacksquare$$

We remark again that since no side conditions have been imposed on $u(x)$, this solution is not unique. Any harmonic function could be added to $u(x)$ and the sum would also satisfy $-\nabla^2 u(x) = f(x)$.

9. Green's Functions for the Laplacian

Throughout this section, U is assumed to be a bounded open, connected set in R^n, whose boundary ∂U is sufficiently smooth that the divergence theorem holds. Consider the Dirichlet boundary value problem for Poisson's equation,

$$-\nabla^2 u(x) = F(x), \qquad for \ x \in U, \qquad and \qquad u(x) = g(x) \qquad for \ x \in \partial U \tag{9.1}$$

We know that

$$u(x) = \int_{R^n} E_n(x-y) F(y)\, dy,$$

satisfies the partial differential equation but this function, does not, in general, satisfy the Dirichlet boundary condition. In order to find a function which satisfies both the equation and the boundary condition, recall that for smooth functions $u(x)$ and $v(x)$

$$\int_U [v(y) \nabla_y^2 u(y) - u(y) \nabla_y^2 v(y)]\ dy = \int_{\partial U} [v(y) \partial_N u(y) - u(y)\, \partial_N v(y)]\, dS(y) \tag{9.2}$$

For x in U fixed but arbitrary, let $v(y) = E_n(x-y) - \phi(y)$ in (9.2) where denotes a yet to be specified function that is harmonic in U. Then since $E_n(x-y)$ is a fundamental solution and

ϕ is harmonic in U,

$$-\int_U u(y)\,\nabla_y^2 v(y) = \int_U u(y)[-\nabla_y^2 E_n(x-y) - 0]\ dy = u(x)$$

Since $u(x)$ solves the Dirichlet problem, (9.2) becomes now,

$$u(x) = -\int_U [v(y)\,\nabla_y^2 u(y)]\ dy + \int_{\partial U}[v(y)\,\partial_N u(y) - u(y)\,\partial_N v(y)]\ dS(y)$$

$$= \int_U v(y)\,F(y)dy - \int_{\partial U} g(y)\partial_N v(y)\,dS(y) + \int_{\partial U} v(y)\,\partial_N u(y)dS(y)$$

If the values of $\partial_N u(y)$ were known on ∂U then this would be an expression for the solution $u(x)$ in terms of the data in the problem. Since $\partial_N u(y)$ on the boundary is not given, we instead choose the harmonic function ϕ in such a way as to make the integral containing this term disappear. Let ϕ be the solution of the following Dirichlet problem,

$$\nabla_y^2 \phi(y) = 0 \quad \text{for } y \in U, \quad \phi(y) = E_n(x-y), \quad \text{for } y \in \partial U$$

where we recall that x denotes some fixed but arbitrary point in U. Then $v(y) = E_n(x-y) - \phi(y) = 0$ on the boundary and the previous expression for $u(x)$ reduces to

$$u(x) = \int_U G(x,y)\,f(y)\,dy - \int_{\partial U} \partial_N G(x,y)\,g(y)\,dS(y) \qquad (9.3)$$

where $G(x,y) = E_n(x-y) - \phi(y)$. Formally, $G(x,y)$ solves

$$-\nabla^2 G(x,y) = -\nabla^2 E_n(x-y) - 0 = \delta(x-y) \quad \text{for } x,y \in U, \qquad (9.4)$$

$$G(x-y) = 0, \quad \text{for } x \in U,\ y \in \partial U$$

and G(x,y) is known as the **Green's function** for the Dirichlet problem for the Laplacian, or, alternatively, as the Green's function of the first kind. Note that if there are two Green's functions then their difference satisfies a completely homogeneous Dirichlet problem. This would seem to imply uniqueness for the Green's function except for the fact that the uniqueness proofs were for the class of functions $C^2(U) \cap C(U)$ and it is not known that $G(x,y)$ is in this class. This point will be cleared up later.

It can be shown rigorously that $G(x,y) = G(y,x)$ *for all* $x,y \in U$. However, a formal demonstration based on (9.4) proceeds as follows. For $x,z \in U,$ (be careful to note that x and y are points in R^n) apply (9.2) with $u(y) = G(y,z)$ *and* $v(y) = G(y,x)$,

$$\int_U [u(y)\,\nabla_y^2 v(y) - v(y)\nabla_y^2 u(y)]\ dy = -\int_U [G(\vec{y},\vec{z})\,\delta(\vec{y}-\vec{x}) - G(\vec{y},\vec{x})\,\delta(\vec{y}-\vec{z})]dz$$

$$\int_{\partial U}[u(y)\,\partial_N v(y) - v(y)\partial_N u(y)]\ dS(y) = \int_{\partial U}[G(\vec{y},\vec{z})\,\partial_N v(\vec{y}) - G(\vec{y},\vec{x})\,\partial_N v(\vec{y})]dS(y) = 0$$

The last integral vanishes because $G(y,z) = G(y,z) = 0,$ *for* $y \in \partial U$. Then (9.2) implies

$$0 = \int_U [G(y,z)\,\delta(y-x) - G(y,x)\,\delta(y-z)]dz = G(x,z) - G(z,x) \quad \text{for all } x,z \in U.$$

This proof will become rigorous when we have developed the generalized function framework in which this argument has meaning.

***Example* 9.1** Let $U = \{(x_1,x_2) \in R^2 : x_2 > 0\}$. The half space is the simplest example of a set having a boundary (i.e., the boundary of the half space is the $x_1 - axis,\ x_2 = 0$) and we will be able to construct the Green's function of the first kind for this simple set. Note that the half space is not a bounded set (having a boundary is not the same as being bounded!). Since $n = 2,$ we write

$$E(\vec{x}-\vec{y}) = -1/2\pi \ \log|\vec{x}-\vec{y}| = -1/2\pi \ \log\sqrt{(x_1-y_1)^2+(x_2-y_2)^2}$$

For $\vec{x} = (x_1,x_2) \in U,$ let $\vec{x}^* = (x_1,-x_2)$ and let

$$v(\vec{y}) = -1/2\pi \ \log|\vec{x}^*-\vec{y}| = -1/2\pi \ \log\sqrt{(x_1-y_1)^2+(x_2+y_2)^2}\,.$$

Then $v = v(\vec{y})$ is a harmonic function of $\vec{y}$ for $\vec{y} \in U.$ Moreover, v reduces to $v(\vec{y}) = E(\vec{x}-\vec{y})$ for $\vec{y} \in \partial U;$ i.e., $v(y_1,0) = E((x_1,x_2)-(y_1,0)).$ Then

$$\begin{aligned} G(\vec{x},\vec{y}) &= -1/2\pi \ [\log|\vec{x}-\vec{y}| - \log|\vec{x}^*-\vec{y}|] = -1/2\pi \ \log[|\vec{x}-\vec{y}|/|\vec{x}^*-\vec{y}|] \\ &= -1/2\pi \ \log\sqrt{\left[(x_1-y_1)^2+(x_2-y_2)^2\right]/\left[(x_1-y_1)^2+(x_2+y_2)^2\right]}\,. \end{aligned} \qquad (9.5)$$

Note that $G(\vec{x},\vec{y}) = 0$ $\mathit{for}\ \vec{y} \in \partial U; \mathit{i.e.}, G((x_1,x_2),(y_1,0)) = 0.$ It is clear from the construction that for each fixed $\vec{x} = (x_1,x_2) \in U,$ $G(\vec{x},\vec{y})$ is a harmonic function of $\vec{y}$ for $\vec{y} \in U.$

Problem 14 Show that for the half-space $U = \{(x_1,x_2) \in R^2 : x_2 > 0\},$

$$\partial_N G(\bar{x},\bar{y})|_{\bar{y}\in\partial U} = \frac{-1}{\pi}\,\frac{x_2}{(x_1-y_1)^2+x_2^2}$$

so that

$$u(x_1,x_2) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x_2}{(x_1-y_1)^2+x_2^2}\ g(y_1)dy_1$$

solves $\nabla^2 u(x_1,x_2) = 0 \quad \mathit{in}\ U, \quad \mathit{and} \quad u(x_1,0) = g(x_1), \quad x_1 \in R.$

***Example* 9.2** Let $U = \left\{(r,\theta) : 0 < r < R,\ |\theta| < \pi\right\} = D_R(0).$ Suppose $u = u(r,\theta)$ satisfies

$$-\nabla^2 u(r,\theta) = 0, \qquad \mathit{in}\ U, \qquad \mathit{and} \qquad u(R,\theta) = g(\theta) \quad \mathit{on} \quad \partial U = \left\{(r,\theta) : r = R,\ |\theta| < \pi\right\}.$$

In an elementary course on PDE's we would show that for **all choices** of the constants, $a_n, b_n,$

$$u(r,\theta) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} r^n[a_n\cos(n\theta) + b_n\sin(n\theta)]$$

solves Laplace's equation in the disc, U. Moreover, the boundary condition is satisfied if

$$u(R,\theta) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} R^n[a_n\cos(n\theta) + b_n\sin(n\theta)] = g(\theta). \qquad (9.6)$$

Then we would appeal to the theory of Fourier series which asserts that any continuous g can be expressed as

$$g(\theta) = \tfrac{1}{2}A_0 + \sum_{n=1}^{\infty}[A_n\cos(n\theta) + B_n\sin(n\theta)] \qquad (9.7)$$

where

$$A_n = 1/\pi \int_{-\pi}^{\pi} g(s)\cos(ns)\,ds, \qquad B_n = 1/\pi \int_{-\pi}^{\pi} g(s)\sin(ns)\,ds.$$

Then, comparing (9.6) with (9.7), it follows that $R^n a_n = A_n,\ R^n b_n = B_n,$ and so

$$u(r,\theta) = \tfrac{1}{2}A_0 + \sum_{n=1}^{\infty}(r/R)^n[A_n\cos(n\theta) + B_n\sin(n\theta)]$$

satisfies both the PDE and the boundary condition. By uniqueness, this must be the solution of the boundary value problem. If we write

$$\begin{aligned} A_n\cos(n\theta) + B_n\sin(n\theta) &= 1/\pi \int_{-\pi}^{\pi} g(s)\cos(ns)\,ds\ \cos(n\theta) + 1/\pi \int_{-\pi}^{\pi} g(s)\sin(ns)\,ds\ \sin(n\theta). \\ &= 1/\pi \int_{-\pi}^{\pi} g(s)[\cos(ns)\cos(n\theta) + \sin(ns)\sin(n\theta)]\ ds \\ &= 1/\pi \int_{-\pi}^{\pi} g(s)\ \cos(n(\theta - s))\ ds, \end{aligned}$$

then $u(r,\theta)$ can be written as

$$\begin{aligned} u(r,\theta) &= 1/\pi \int_{-\pi}^{\pi}[\tfrac{1}{2} + \sum_{n=1}^{\infty}(r/R)^n\ \cos(n(\theta - s))]\,g(s)\,ds, \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - s) + r^2}\,g(s)\,ds \end{aligned}$$

Here the series in n was summed by writing $\cos(n(\theta - s))$ in terms of exp$[\pm in(\theta - s)]$ and recognizing that the series is a geometric series. Then

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - s) + r^2}\,g(s)\,ds = \int_{-\pi}^{\pi}\partial_N G((r,\theta),(R,s))\,g(s)\,ds$$

where $G((r,\theta),(R,s))$ denotes the Green's function for this problem. This representation is often called the ***Poisson integral formula***.

10. The Inverse Laplace Operator

We are all familiar with problems of the form $A\vec{x} = \vec{f}$ where A denotes an n by n matrix and $\vec{x},\vec{f}$ denote vectors in the linear space R^n. In this situation, A can be viewed as a linear operator from the linear space R^n into R^n. If the only solution of $A\vec{x} = \vec{0},\ is\ \vec{x} = \vec{0},$ then $A\vec{x} = \vec{f}$ has a unique solution $\vec{x}$ for every data vector $\vec{f}$. This solution can be expressed as $\vec{x} = A^{-1}\vec{f},$ where A^{-1} denotes the inverse of the matrix A. There are strong analogies between the problem $A\vec{x} = \vec{f}$ on R^n and the problem (9.1).

Consider problem (9.1) in the special case $g = 0$; i.e.,

$$-\nabla^2 u(\vec{x}) = f(x), \quad x \in U, \quad u(x) = 0, \quad x \in \partial U. \qquad (10.1)$$

Recall that we showed that the only solution of (9.1) when $g = f = 0,$ is $u = 0,$ so the solution to (10.1) is unique.
In fact, the unique solution $u = u(\vec{x})$, can be expressed in terms of the Green's function by

$$u(\vec{x}) = \int_U G(\vec{x},\vec{y})\,f(\vec{y})\,d\vec{y} \qquad (10.2)$$

If we define $K[f](\vec{x}) = \int_U G(\vec{x},\vec{y})\, f(\vec{y})\, d\vec{y} \quad \text{for any } f \in C(\bar{U}),$

then it is clear that

$$K[C_1 f_1 + C_2 f_2] = C_1 K[f_1] + C_2 K[f_2] \quad \text{for all } f_1, f_2 \in C(U),\ C_1, C_2 \in R.$$

We say that K is a ***linear operator*** on the ***linear space*** $C(\bar{U})$. We recall that to say that $C(\bar{U})$ is a linear space is to say that for all $f_1, f_2 \in C(\bar{U})$ and for all $C_1, C_2 \in R.$, the linear combination $C_1 f_1 + C_2 f_2$ is also in $C(\bar{U})$.

The problem (10.1) can be expressed in operator notation. Define an operator L by

$$L[u](\vec{x}) = -\nabla^2 u(\vec{x}) \quad \text{for any } \ u \in D = \left\{ u \in C^2(\bar{U}) : u(\vec{x}) = 0 \ \text{ for } \vec{x} \in \partial U \right\}.$$

Then for any $u \in D$ it follows that $L[u](\vec{x}) \in C(\bar{U})$ so L can be viewed as a function defined on D with values in $C(\bar{U})$. Since D is a subspace of $C(\bar{U})$ we can even say that L is a function from $C(\bar{U})$ into $C(\bar{U})$ but we should note that L is not defined on all of $C(\bar{U})$.
It is also easy to check that L is a linear operator from D into $C(\bar{U})$, and (10.1) can be expressed in terms of this linear operator as follows,

$$\text{find } \ u \in D \quad \text{such that } \ L[u] = f \ \in C(\bar{U}).$$

The uniqueness for (10.1), stated in the operator terminology, becomes

$$L[u] = 0 \ \text{ if and only if } \ u = 0.$$

Evidently, the operators K and L are related by,

$$a)\ K[f](\vec{x}) \in D \quad \text{for any } f \in C(\bar{U}), \text{ and } \ L[K[f](\vec{x})] = f(\vec{x})$$

$$b)\ \text{for any } \ u \in D,\ L[u](\vec{x}) \in C(\bar{U}), \quad \text{and} \quad K[L[u](\vec{x})] = u(\vec{x}).$$

These two statements together assert that $K = L^{-1}$, K is the operator inverse to L.

If we use the notation $\langle \vec{x}, \vec{z} \rangle$ to denote the usual inner product between two vectors $\vec{x}, \vec{z}$, then

$$\langle A\vec{x}, \vec{z} \rangle = \langle \vec{x}, A^\top \vec{z} \rangle \qquad \text{for all } \ \vec{x}, \vec{z} \in R^n.$$

Here $A^\top$ denotes the matrix transpose of A. It is a fact from linear algebra that the dimension of the null space of the matrix A is equal to the dimension of the null space of the transpose matrix, $A^\top$. If the null space of A has positive dimension then the solution of $A\vec{x} = \vec{f}$ is not unique. What is more, if $\vec{z}$ denotes any vector in the null space of $A^\top$ then

$$\left\langle \vec{f}, \vec{z} \right\rangle = \langle A\vec{x}, \vec{z} \rangle = \langle \vec{x}, A^\top \vec{z} \rangle = 0$$

and it is then evident that a necessary condition for the existence of a solution for $A\vec{x} = \vec{f}$ is that $\left\langle \vec{f}, \vec{z} \right\rangle = 0$ for all $\vec{z}$ in the null space of $A^\top$. The matrix A is said to be ***symmetric*** if either of the following equivalent conditions applies, $A = A^\top \ \text{ or } \ \langle A\vec{x}, \vec{z} \rangle = \langle \vec{x}, A\vec{z} \rangle$ for all $\vec{x}, \vec{z}$. When A is symmetric, the null space of A not only has the same dimension as that of $A^\top$, the two null spaces are actually the same. In this case, $A\vec{x} = \vec{f}$ has no solution unless $\left\langle \vec{f}, \vec{z} \right\rangle = 0$ for all $\vec{z}$ in the null space of A. If this condition is satisfied, then any two solutions of $A\vec{x} = \vec{f}$

differ by an element from the null space of A.

We will now consider the analogue of these last results for the case of a boundary value problem for Laplace's equation. First, we have to have an inner product on the function space $C(\bar{U})$. The essential properties of the inner product are

$i)\ \langle \vec{x},\vec{z}\rangle = \langle \vec{z},\vec{x}\rangle$ for all $\vec{x},\vec{z}.$

$ii)\ \langle C\vec{x},\vec{z}\rangle = C\langle \vec{x},\vec{z}\rangle$ for all $\vec{x},\vec{z},\ and\ all\ \ C \in R.$

$iii)\ \langle \vec{x}+\vec{y},\vec{z}\rangle = \langle \vec{x},\vec{z}\rangle + \langle \vec{y},\vec{z}\rangle$ for all $\vec{x},\vec{y},\vec{z}.$

$iv)\ \langle \vec{x},\vec{x}\rangle \geq 0$ for all $\vec{x},\ and\ \ \langle \vec{x},\vec{x}\rangle = 0,\ if\ and\ only\ if\ \vec{x} = \vec{0}.$

and any mapping from $R^n \times R^n\ to\ R$ having these four properties is called an inner product on the linear space R^n.

We can define an ***inner product*** on the function space $C(\bar{U})$, by letting

$$\left\langle f_1,f_2 \right\rangle = \int_U f_1(\vec{x})f_2(\vec{x})\,d\vec{x} \qquad for\ all\ f_1,f_2 \in C(\bar{U}).$$

This is just a generalization of the vector inner product for vectors on R^n and it is easy to check that the four properties given above are all satisfied for this product.

We observe now, that

$$\left\langle K[f_1],f_2 \right\rangle = \int_U K[f_1](\vec{x})\,f_2(\vec{x})\,d\vec{x} = \int_U \int_U G(\vec{x},\vec{y})f_1(\vec{y})\,d\vec{y}\,f_2(\vec{x})\,d\vec{x} \qquad for\ all\ f_1,f_2 \in C(\bar{U}).$$

Note further that

$$\int_U \int_U G(\vec{x},\vec{y})f_1(\vec{y})\,d\vec{y}\,f_2(\vec{x})\,d\vec{x} = \int_U \int_U G(\vec{x},\vec{y})\,f_2(\vec{x})\,d\vec{x}\,f_1(\vec{y})\,d\vec{y} = \langle f_1,K^{\top}[f_2]\rangle$$

where $K^{\top}[f_2]$ is defined by

$$K^{\top}[f] = \int_U G(\vec{x},\vec{y})\,f(\vec{x})\,d\vec{x} \qquad for\ any\ f \in C(\bar{U}).$$

Clearly, $K^{\top}[f]$ defines another linear operator on $C(U)$. When

$$\left\langle K[f_1],f_2 \right\rangle = \langle f_1,K^{\top}[f_2]\rangle\ for\ all\ f_1,f_2 \in C(U),$$

we say that $K^{\top}$ is the ***adjoint*** of the operator K. Since we know that

$$G(\vec{x},\vec{y}) = G(\vec{y},\vec{x}) \text{ for all } \vec{x},\vec{y} \in U,$$

it follows that $\qquad K[f] = K^{\top}[f]\ \ for\ any\ f \in C(\bar{U}).$

We say that the operator K is ***symmetric***. Since

$$\left\langle K[f_1],f_2 \right\rangle = \langle f_1,K^{\top}[f_2]\rangle \qquad for\ all\ f_1,f_2 \in C(\bar{U}),$$

and $K = L^{-1},$ it seems reasonable to expect that $\langle L[u],v\rangle = \langle u,L[v]\rangle\ \ for\ all\ \ u,v \in D$. That

this is, in fact, the case follows from (3.4).
That is,

$$\langle L[u], v\rangle = \int_U -v\nabla^2 u\, dx = \int_U -u\nabla^2 v\, dx - \int_{\partial U}(v\partial_N u - u\partial_N v)\, dS$$

$$= \int_U -u\nabla^2 v\, dx = \langle u, L[v]\rangle \qquad for\ all \quad u,v \in D.$$

Now consider the Neumann problem

$$-\nabla^2 u(\vec{x}) = f(x), \qquad x \in U, \qquad \partial_N u(x) = 0, \qquad x \in \partial U. \tag{10.3}$$

Problem (10.3) can be expressed in terms of the following operator,

$$L_N[u](\vec{x}) = -\nabla^2 u(\vec{x}) \qquad for\ any \quad u \in D_N = \left\{u \in C^2(\bar{U}) : \partial_N u(\vec{x}) = 0 \quad for\ \vec{x} \in \partial U\right\}.$$

as

$$find \quad u \in D_N \quad such\ that \quad L_N[u] = f \ \in C(\bar{U}).$$

Although the action of this operator, $L_N[u]$, is the same as that of the previously defined operator, L, it is not the same operator since they have different domains. In particular, D_N contains all constant functions and these functions belong to the null space of L_N. Then L_N is not invertible. However, the same argument used above shows that L_N. is symmetric. Then $L_N[u] = f$ has no solution unless f satisfies $\langle f, v\rangle = 0$ for all constant functions v. If this condition is satisfied, then any two solutions differ by a constant. This fact was already mentioned in the beginning of section 7 but now we see it in a new setting. It is just the analogue of the linear algebra result for singular matrices A.