Laplace Transform

Definition of the Transform

Starting with a given function of t, $f(t)$, we can define a new function $\hat{f}(s)$ of the variable s. This new function will have several properties which will turn out to be convenient for purposes of solving linear constant coefficient ODE's. The definition of $\hat{f}(s)$ is as follows:

Definition Let $f(t)$ be defined for $t \geq 0$ and let the Laplace transform of $f(t)$ be defined by,

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt = \hat{f}(s)$$

For example:

$$f(t) = 1,\ \forall t \geq 0, \qquad L[1] = \int_0^\infty e^{-st}\, dt = \frac{e^{-st}}{-s}\Big|_{t=0}^{t=\infty} = \frac{1}{s} = \hat{f}(s) \quad for\ s > 0$$

$$f(t) = e^{bt},\ \forall t \geq 0, \qquad L[e^{bt}] = \int_0^\infty e^{-(b-s)t}\, dt = \frac{e^{-(b-s)t}}{(s-b)}\Big|_{t=0}^{t=\infty} = \frac{1}{s-b} = \hat{f}(s), \quad for\ s > b.$$

The Laplace transform is defined for all functions of **exponential type**. That is, any function $f(t)$ which is

(a) piecewise continuous = has at most finitely many finite jump discontinuities on any interval of finite length

(b) has exponential growth: for some positive constants M and k

$$|f(t)| \leq Me^{kt} \quad for\ all\ t \geq 0, \quad .$$

Properties of the Laplace Transform

The Laplace transform has the following **general properties**:

1. **Linearity** $\quad L[C_1 f(t) + C_2 g(t)] = C_1\hat{f}(s) + C_2\hat{g}(s)$

2. **Homogeneity** $\quad L[f(at)] = \frac{1}{a}\hat{f}\left(\frac{s}{a}\right) \quad for \quad a > 0$

3. **Transform of the Derivative** $\quad L[f'(t)] = s\hat{f}(s) - f(0)$

$$L[f''(t)] = s^2\hat{f}(s) - sf(0) - f'(0) \quad etc$$

4. **Derivative of the Transform** $\quad L[tf(t)] = -\hat{f}'(s)$

$$L[t^2 f(t)] = (-1)^2\hat{f}''(s) \quad etc$$

To see that 3 is true, write

$$L[f'(t)] = \int_0^\infty e^{-st} f'(t)\,dt = e^{-st} f(t)\big|_{t=0}^{t=\infty} - \int_0^\infty (-s\,e^{-st}) f(t)\,dt$$

$$= -f(0) + s\int_0^\infty e^{-st} f(t)\,dt = s\hat{f}(s) - f(0),$$

where we used integration by parts on the first integral. Since the derivative of $f'(t)$ is $f''(t)$, we can apply this result to $f'(t)$, to get

$$L[f''(t)] = s\,L[f'(t)] - f'(0) = s\left\{s\hat{f}(s) - f(0)\right\} - f'(0) = s^2\hat{f}(s) - sf(0) - f'(0)$$

To show 4, just note that

$$\frac{d}{ds}\hat{f}(s) = \frac{d}{ds}\int_0^\infty e^{-st} f(t)\,dt = \int_0^\infty \frac{d}{ds} e^{-st} f(t)\,dt = \int_0^\infty -t\,e^{-st} f(t)\,dt = L[-tf(t)].$$

Application of the Properties

Now these two results can be applied to derive more Laplace transform formulas.

e.g., $L[t] = L[t \bullet 1] = -\frac{d}{ds}\left(\frac{1}{s}\right) = \frac{1}{s^2}$

$$L[t^2] = L[t \bullet t] = -\frac{d}{ds}\left(\frac{1}{s^2}\right) = \frac{2}{s^3}$$

$$L[t^3] = L[t \bullet t^2] = -\frac{d}{ds}\left(\frac{2}{s^3}\right) = \frac{6}{s^4}$$

and, more generally, $L[t^n] = \frac{n!}{s^{n+1}},\ n = 0, 1, 2, \ldots$

We can also apply this trick to find the transform of e^{bt},

$$L[te^{bt}] = -\frac{d}{ds}\left(\frac{1}{s-b}\right) = \frac{1}{(s-b)^2}.$$

In addition, for $f(t) = \sin t,\ f'(t) = \cos t,\ f''(t) = -\sin t,$ hence 3b implies

$$L[-\sin t] = s^2\hat{f}(s) - sf(0) - f'(0) = s^2\hat{f}(s) - 1$$

and

$$L[-\sin t] = -\hat{f}(s)$$

so

$$-\hat{f}(s) = s^2\hat{f}(s) - 1, \quad i.e., \qquad \hat{f}(s) = \frac{1}{s^2+1}.$$

Similarly, if $g(t) = \cos t,\ g'(t) = -\sin t,\ g''(t) = -\cos t,$ then

$$L[-\cos t] = s^2\hat{g}(s) - sg(0) - g'(0) = s^2\hat{g}(s) - s$$

and

$$L[-\cos t] = -\hat{g}(s).$$

Then

$$-\hat{g}(s) = s^2\hat{g}(s) - s, \quad i.e., \qquad \hat{g}(s) = \frac{s}{s^2+1}.$$

Furthermore, property 2 implies that

$$L[\sin \omega t] = \frac{1}{\omega}\hat{f}\left(\frac{s}{\omega}\right) = \frac{1}{\omega}\frac{1}{\left(\frac{s}{\omega}\right)^2+1}. = \frac{\omega}{s^2+\omega^2}.$$

and

$$L[\cos \omega t] = \frac{1}{\omega}\hat{g}\left(\frac{s}{\omega}\right) = \frac{1}{\omega}\frac{\frac{s}{\omega}}{\left(\frac{s}{\omega}\right)^2+1}. = \frac{s}{s^2+\omega^2}.$$

In addition

$$L[t\sin\omega t] = -\frac{d}{ds}\left(\frac{\omega}{s^2+\omega^2}\right) = \frac{2\omega s}{(s^2+\omega^2)^2}$$

and

$$L[t\cos\omega t] = -\frac{d}{ds}\left(\frac{s}{s^2+\omega^2}\right) = \frac{s^2-\omega^2}{(s^2+\omega^2)^2}.$$

Initial Value Problems

The Laplace transform can be used to solve initial value problems for linear differential equations having constant coefficients. For example, consider

$$y'(t) + ky(t) = 5e^{-kt}, \qquad y(0) = A.$$

If we let $\hat{y}(s)$ denote the Laplace transform of the solution, $y(t)$, then

$$s\hat{y}(s) - y(0) + k\hat{y}(s) = (s+k)\hat{y}(s) - A = \frac{5}{s+k}$$

Then

$$\hat{y}(s) = \frac{A}{s+k} + \frac{5}{(s+k)^2}.$$

Now we observe that

$$L^{-1}\left[\frac{A}{s+k}\right] = AL^{-1}\left[\frac{1}{s+k}\right] = Ae^{-kt}$$

and

$$L^{-1}\left[\frac{5}{(s+k)^2}.\right] = 5L^{-1}\left[\frac{1}{(s+k)^2}.\right] = 5te^{-kt}$$

hence

$$y(t) = Ae^{-kt} + 5te^{-kt}.$$

Additional Properties

Let $f(t)$ be a function of exponential type and suppose that for some $b > 0$,

$$h(t) = \left\{ \begin{array}{ll} 0 & if \;\; 0 < t < b \\ f(t-b) & if \;\; t > b \end{array} \right\}$$

Then $h(t)$ is just the function $f(t)$, delayed by the amount b. Then

$$L[h(t)] = \int_0^\infty h(t)\, e^{-st}\, dt = \int_b^\infty f(t-b)\, e^{-st}\, dt$$

Let $z = t - b$ so that

$$L[h(t)] = \int_0^\infty f(z)\, e^{-s(z+b)}\, dz = e^{-bs}\int_0^\infty f(z)\, e^{-sz}\, dz = e^{-bs}\hat{f}(s).$$

If we define $H(t-b) = \left\{ \begin{array}{ll} 0 & if \;\; 0 < t < b \\ 1 & if \;\; t > b \end{array} \right\}$

then

$$h(t) = H(t-b) f(t-b)$$

and we find

5. **Transform of a Delay** $\quad L[H(t-b)f(t-b)] = e^{-bs}\hat{f}(s), \;\; for \;\; b > 0.$

A related results is the following

$$L[e^{bt}f(t)] = \int_0^\infty e^{bt} f(t)\, e^{-st}\, dt = \int_0^\infty f(t)\, e^{-(s-b)t}\, dt = \hat{f}(s-b).$$

i.e.,

6. **Delay of a Transform** $\quad L[e^{bt}f(t)] = \hat{f}(s-b)$

Results 5 and 6 assert that a delay in the function induces an exponential multiplier in the transform and, conversely, a delay in the transform is associated with an exponential multiplier for the function.

A final property of the Laplace transform asserts that

7. **Inverse of a Product** $L[(f*g)(t)] = \hat{f}(s)\hat{g}(s)$

$$\textit{where} \quad (f*g)(t) := \int_0^t f(t-\tau)g(\tau)\,d\tau$$

The product, $(f*g)(t)$, is called the **convolution product** of f and g. Life would be simpler if the inverse Laplace transform of $\hat{f}(s)\hat{g}(s)$ was the pointwise product $f(t)g(t)$, but it isn't, it is the convolution product. The convolution product has some of the same properties as the pointwise product, namely

$$(f*g)(t) = (g*f)(t) \qquad \textit{and} \qquad (h*(f*g))(t) = ((h*f)*g)(t).$$

We will not give the proof of the result 7 but will make use of it nevertheless.

Applications of the Properties

We can use property 6 together with the results

$$L[\sin\omega t] = \frac{\omega}{s^2+\omega^2} \qquad \text{and} \qquad L[\cos\omega t] = \frac{s}{s^2+\omega^2}$$

to derive the formulas

$$L[e^{bt}\sin\omega t] = \frac{\omega}{(s-b)^2+\omega^2} \qquad \text{and} \qquad L[e^{bt}\cos\omega t] = \frac{s-b}{(s-b)^2+\omega^2}$$

Now consider the problem

$$y''(t) + 2y'(t) + 10y(t) = 1, \qquad y(0) = y'(0) = 0.$$

Transforming this problem leads to

$$(s^2+2s+10)\hat{y}(s) = \frac{1}{s},$$

and

$$\hat{y}(s) = \frac{1}{s}\frac{1}{s^2+2s+10} = \frac{1}{s}\frac{1}{(s+1)^2+9} = \hat{f}(s)\hat{g}(s).$$

We know that $L^{-1}\left[\frac{1}{s}\right] = 1,\ \textit{and}\ L^{-1}\left[\frac{3}{(s+1)^2+9}\right] = e^{-t}\sin 3t.$ Then by property 7,

$$y(t) = (f*g)(t) := \int_0^t f(t-\tau)g(\tau)\,d\tau = \int_0^t 1(t-\tau)e^{-\tau}\sin 3\tau\,d\tau$$

$$= \int_0^t e^{-\tau}\sin 3\tau\,d\tau = \tfrac{3}{10} - \tfrac{3}{10}e^{-t}\cos(3t) - \tfrac{1}{10}e^{-t}\sin(3t).$$

Now consider the problem where the forcing is changed to a piecewise constant function,

$$y''(t) + 2y'(t) + 10y(t) = f(t) = \left\{ \begin{array}{lll} 1 & if & 0 < t < 1 \\ 2 & if & 1 < t < 2 \\ 3 & if & 2 < t \end{array} \right\}, \qquad y(0) = y'(0) = 0.$$

First, we observe that $f(t) = H(t) + H(t-1) + H(t-2)$, and so, by property 5,

$$\hat{f}(s) = \frac{1}{s} + \frac{1}{s}e^{-s} + \frac{1}{s}e^{-2s}.$$

Then

$$\hat{y}(s) = \left(\frac{1}{s} + \frac{1}{s}e^{-s} + \frac{1}{s}e^{-2s}\right)\frac{1}{s^2 + 2s + 10}$$

$$= \left(\frac{1}{s} + \frac{1}{s}e^{-s} + \frac{1}{s}e^{-2s}\right)\frac{1}{(s+1)^2 + 9} = \hat{f}(s)\,\hat{g}(s).$$

However, we already know that

$$L^{-1}\left[\frac{1}{s}\,\frac{3}{(s+1)^2 + 9}\right] = \frac{3}{10} - \frac{3}{10}\,e^{-t}\cos(3t) - \frac{1}{10}\,e^{-t}\sin(3t) := P(t)$$

hence it follows from property 5 that,

$$L^{-1}\left[\frac{1}{s}e^{-s}\,\frac{3}{(s+1)^2 + 9}\right] = H(t-1)\,P(t-1)$$

$$L^{-1}\left[\frac{1}{s}e^{-2s}\,\frac{3}{(s+1)^2 + 9}\right] = H(t-2)\,P(t-2)$$

and $\quad y(t) = P(t) + H(t-1)\,P(t-1) + H(t-2)\,P(t-2).$

A plot of this solution is shown below.

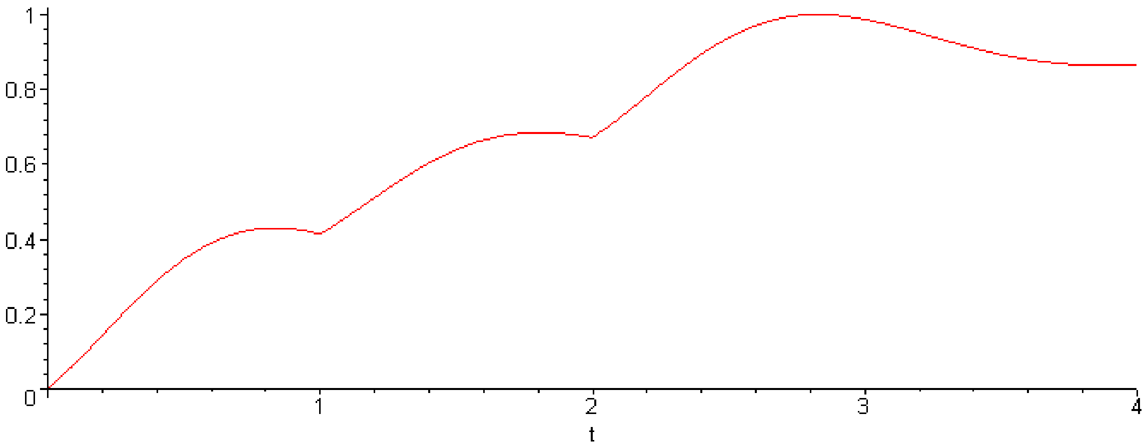

Table of Transform Properties

1. $L[C_1 f(t) + C_2 g(t)] = C_1 \hat{f}(s) + C_2 \hat{g}(s)$

2. $L[f(at)] = \frac{1}{a}\hat{f}\left(\frac{s}{a}\right) \quad for \quad a > 0$

3. $L[f'(t)] = s\hat{f}(s) - f(0)$

   $L[f''(t)] = s^2\hat{f}(s) - sf(0) - f'(0) \quad etc$

4. $L[tf(t)] = -\hat{f}'(s)$

   $L[t^2 f(t)] = (-1)^2\hat{f}''(s) \quad etc$

5. $L[H(t-b)f(t-b)] = e^{-bs}\hat{f}(s), \quad for \quad b > 0.$

6. $L[e^{bt}f(t)] = \hat{f}(s-b)$

7. $L[(f*g)(t)] = \hat{f}(s)\hat{g}(s) \quad where \quad (f*g)(t) := \int_0^t f(t-\tau)g(\tau)\,d\tau$

Table of Laplace Transform Formulas

$$L[1] = \frac{1}{s} \quad for\ s > 0$$

$$L[e^{bt}] = \frac{1}{s-b} \quad for\ s > b.$$

$$L[e^{at} - e^{bt}] = \frac{a-b}{(s-a)(s-b)} \quad for\ s > b.$$

$$L[t^n] = \frac{n!}{s^{n+1}},\ n = 0, 1, 2, ...$$

$$L[te^{bt}] = \frac{1}{(s-b)^2}.$$

$$L[\sin \omega t]. = \frac{\omega}{s^2 + \omega^2}.$$
$$L[\cos \omega t]. = \frac{s}{s^2 + \omega^2}.$$

$$L[t \sin \omega t] = \frac{2\omega s}{(s^2 + \omega^2)^2}$$
$$L[t \cos \omega t] = \frac{s^2 - \omega^2}{(s^2 + \omega^2)^2}.$$

$$L[e^{bt} \sin \omega t] = \frac{\omega}{(s-b)^2 + \omega^2}$$

$$L[e^{bt} \cos \omega t] = \frac{s-b}{(s-b)^2 + \omega^2}$$