Solutions of First Order Ordinary Differential Equations

Separable equations

Perhaps the simplest case in which the solution can be constructed is when the differential equation has the form

$$\frac{dy}{dt} = P(y)\, Q(t)$$

We say the equation is *separable* and proceed to integrate by writing

$$\int \frac{1}{P(y)} \frac{dy}{dt} dt = \int \frac{dy}{P(y)} = \int Q(t) dt$$

For example, if

$$\frac{dy}{dt} = ty(t)$$

then

$$\int \frac{dy}{y} = \int t dt \qquad \text{or} \qquad \ln y(t) = t^2/2 + C_1$$

and

$$y(t) = C_2 e^{t^2/2}.$$

Sometimes the integration is more difficult as in the following example,

$$\frac{dy}{dt} = -y(t)(1 - y(t))$$

Then

$$\int \frac{dy}{y(1-y)} = -\int dt$$

We must employ partial fractions to find

$$\int \frac{dy}{y(1-y)} = \ln y - \ln(y-1) = \ln \frac{y}{y-1} = -t + C_1.$$

Then

$$\frac{y(t)}{y(t)-1} = C_2 e^{-t} \qquad \text{or} \qquad y(t) = C_2 e^{-t}[y(t) - 1].$$

Finally the explicit solution can be written as,

$$y(t) = \frac{C_2 e^{-t}}{C_2 e^{-t} - 1}.$$

Exact Equations

Another rather simple case occurs when the equation takes the form of an exact differential. To see what this means suppose $F(t, y(t))$ is a smooth function of two variables. Then the total differential is written,

$$dF = \frac{\partial F}{\partial t} dt + \frac{\partial F}{\partial y} dy.$$

Then it is clear that the differential equation

$$\frac{\partial F}{\partial t} dt + \frac{\partial F}{\partial y} dy = 0 \qquad \text{or} \qquad \frac{\partial F}{\partial y} \frac{dy}{dt} = -\frac{\partial F}{\partial t}$$

is equivalent to $F(t, y) =$ constant. For example, $F(t, y) = y^p t^q$ leads to the equation

$$q t^{q-1} y^p \, dt + p y^{p-1} t^q \, dy = 0$$

whose solution is $y^p t^q = C$ or $y(t) = C t^{q/p}$. In a case like this, we say the differential equation is *exact*. In general, the equation

$$P(y, t) dy + Q(y, t) dt = 0$$

is exact if $P = \frac{\partial F}{\partial y}$ and $Q = \frac{\partial F}{\partial t}$ for some smooth function $F(t, y)$. If such an F exists then it is necessary that

$$\frac{\partial}{\partial t}\left(\frac{\partial F}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial t}\right) \qquad \text{or} \qquad \frac{\partial P}{\partial t} = \frac{\partial Q}{\partial y}$$

For example, consider $(2t - 2y)dy - (4t - 2y + 5)dt = 0.$

Then $P(y, t) = 2t - 2y$ and $\frac{\partial P}{\partial t} = 2,$ $Q(y, t) = 4t - 2y + 5$ *and* $\frac{\partial Q}{\partial y} = 2.$
This tells us there exists a smooth $F(t, y)$ for which

$$P = \frac{\partial F}{\partial y} = 2t - 2y \qquad \text{and} \qquad Q = \frac{\partial F}{\partial t} = -(4t - 2y + 5)$$

Integrating the first of these equalities with respect to y implies $F(t, y) = 2ty - y^2 + \phi(t),$

while integrating the second equality with respect to t leads to
$F(t, y) = -2t^2 + 2ty - 5t + \psi(y).$

Now, using the first result, we compute $\frac{\partial F}{\partial t} = 2y + \phi'(t) = -4t + 2y - 5.$

Then $\phi'(t) = -4t - 5$ and $\phi(t) = -2t^2 - 5t + C_1$

Similarly, the second result implies $\dfrac{\partial F}{\partial y} = 2t + \psi'(y) = 2t - 2y$

and $\psi'(y) = -2y,$ pr $\psi(y) = -y^2 + C_2.$ Then combining these results leads to,

$$F(t,y) = 2ty - y^2 - 2t^2 - 5t + C_1 = -2t^2 + 2ty - 5t + -y^2 + C_2$$

and the implicit solution to the differential equation can be written

$$y^2 - 2ty + 2t^2 + 5t = C_3.$$

Solution Methods for Linear Equations

The most general first order linear ODE is an equation of the form

$$p(t)\frac{dy}{dt} + q(t)y(t) = f(t). \qquad (1)$$

Here p and q are called **coefficients** and f is referred to as the **forcing term** in the equation. When f=0, we say the equation is **homogeneous** and when f is not identically zero, we say the equation is **inhomogeneous**.

It is usually customary to divide equation (1) by the coefficient of the derivative so as to have the equation is so called **normal form**,

i.e., $y'(t) + \dfrac{q(t)}{p(t)}y(t) = \dfrac{f(t)}{p(t)}$ becomes $y'(t) + a(t)\,y(t) = F(t). \qquad (2)$

The most direct way to solve (2) is to suppose $A(t)$ is an anti-derivative of $a(t),$ that is, $A'(t) = a(t)$ and to note that

$$\frac{d}{dt}(e^{A(t)}y(t)) = e^{A(t)}y'(t) + A'(t)e^{A(t)}y(t) = e^{A(t)}[y'(t) + a(t)y(t)].$$

Then $\dfrac{d}{dt}(e^{A(t)}y(t)) = e^{A(t)}F(t)$

and $e^{A(t)}y(t) = \int^t e^{A(s)}F(s)ds$ or $y(t) = e^{-A(t)}\int^t e^{A(s)}F(s)ds.$

While this is certainly an efficient approach to solving the differential equation, it conceals some of the structure of the solution. For this reason we will consider some other approaches, less efficient in the short run but more illuminating in the long run.

Solutions of Inhomogeneous Equations

In order to solve equation (2) in the general inhomogeneous case (i.e., $F \neq 0$) we will first solve the corresponding homogeneous version of $(2),$ that is,

$$y'(t) + a(t)\, y(t) = 0.$$

We separate and integrate $$\int \frac{dy}{y} = \ln y = -\int^t a(s)\, ds + C_0 = -A(t) + C_0$$

and find the homogeneous solution to be $$y_H(t) = C_1 e^{-A(t)} \quad where \quad A'(t) = a(t).$$

Here $y_H(t)$ is referred to as the **general solution** for the homogeneous equation. This means that for any choice of the constant C_1, the function $y_H(t)$ solves the homogeneous ODE, and every solution of the homogeneous equation must be given by this formula for some choice of C_1. The first half of this statement has been proved by the construction we have just performed. The second half of the statement is true but has not been proved yet. At any rate, we will now use the homogeneous solution to find a solution for the inhomogeneous equation. This is accomplished by supposing the inhomogeneous equation has a solution of the form

$$y_p(t) = C(t)\, e^{-A(t)}.$$

We refer to this as a **particular solution**. Here $C(t)$ denotes an unknown function which we will now find. Note that

$$y_p'(t) = C'(t)\, e^{-A(t)} + C(t)\, e^{-A(t)}(-A'(t)).$$

$$= C'(t)\, e^{-A(t)} - C(t)\, e^{-A(t)} a(t).$$

Then $$y_p{}'(t) + a(t)\, y_p(t) = C'(t)\, e^{-A(t)} - C(t)\, e^{-A(t)} a(t) + a(t)\, C(t)\, e^{-A(t)} = C'(t)\, e^{-A(t)},$$

and the inhomogeneous equation now becomes,

$$y_p{}'(t) + a(t)\, y_p(t) = C'(t)\, e^{-A(t)} = F(t)$$

or

$$C'(t) = F(t) e^{A(t)}.$$

We integrate to get $C(t)$,

$$C(t) = \int^t F(\tau) e^{A(\tau)}\, d\tau,$$

and then $$y_p(t) = C(t)\, e^{-A(t)} = e^{-A(t)} \int^t F(\tau) e^{A(\tau)}\, d\tau.$$

This method of finding a particular solution is called the method of **variation of parameters**. The assumption that the particular solution could be written in the form $y_p(t) = C(t) y_H(t)$ had the effect of reducing the inhomogeneous equation to a simple integration to find $C(t)$.

The general solution for the inhomogeneous equation is defined to be the sum of the general homogeneous solution and a particular solution,

$$y(t) = y_H(t) + y_p(t) = C_1 e^{-A(t)} + e^{-A(t)} \int^t F(\tau) e^{A(\tau)} d\tau.$$

Note that if we add any homogeneous solution to a particular solution, we obtain a new particular solution. It is for this reason that we refer to $y_p(t)$ as A particular solution and not THE particular solution.

We will illustrate this method with an example. Consider

$$y'(t) + ky(t) = t \qquad k = \text{constant}$$

We find the homogeneous solution to be

$$y_H(t) = Ce^{-kt}.$$

Now we suppose the particular solution has the following form

$$y_p(t) = C(t)\, e^{-kt}$$

where $\quad C'(t)\, e^{-kt} = F(t) = t$

Then

$$C'(t) = F(t) e^{kt} = t e^{kt},$$
$$C(t) = \int t e^{kt} dt = \frac{1}{k^2} e^{kt}(kt - 1),$$

and

$$y_p(t) = C(t)\, e^{-kt} = \frac{1}{k^2}(kt - 1)$$

Then the general solution for this example is

$$y(t) = Ce^{-kt} + \frac{1}{k^2}(kt - 1).$$

Still another way to obtain the particular solution is by guessing (or we could call it the method of undetermined coefficients). Here we note that since

$$y_p'(t) + ky_p(t) = t,$$

it is reasonable to assume that $y_p(t) = at + b$ where a,b are to be determined. Substituting into the equation, we find

$$(at + b)' + k(at + b) = a + kb + akt = t.$$

Then, equating coefficients of like powers of t on the two sides of this last equation,

$$ak = 1 \qquad \textit{and} \qquad a + kb = 0,$$

or

$$a = \frac{1}{k} \qquad and \qquad b = -\frac{a}{k} = -\frac{1}{k^2}.$$

This leads to $y_p(t) = at + b = \frac{1}{k}t - \frac{1}{k^2}$, which agrees with the previous result.

As a second example, consider the inhomogeneous equation,

$$y'(t) + ky(t) = F_0 \cos\Omega t$$

where k, Ω, F_0 all denote given constants. If we use the method of variation of parameters we suppose $y_p(t) = C(t)\, e^{-kt}$ where

$$C'(t)\, e^{-kt} = F(t) = F_0 \cos\Omega t$$

and $$C'(t) = F(t)e^{kt} = e^{kt}F_0 \cos\Omega t.$$

Then $$C(t) = F_0 \int e^{kt} \cos\Omega t\, dt =: F_0 \frac{e^{kt}}{k^2+\Omega^2}[k \cos\Omega t + \Omega \sin\Omega t],$$

and $$y_p(t) = C(t)\, e^{-kt} = \frac{F_0}{k^2+\Omega^2}[k \cos\Omega t + \Omega \sin\Omega t].$$

This particular solution could as easily have been obtained by guessing. Since the forcing term involves $\cos\Omega t$, it is logical to suppose that the particular solution could only be composed of some combination of $\cos\Omega t$ and $\sin\Omega t$; i.e.,

$$y_p(t) = a \cos\Omega t + b \sin\Omega t \qquad \text{for some constants a,b.}$$

Then $$y_p'(t) + ky_p(t) = \tfrac{d}{dt}(a \cos\Omega t + b \sin\Omega t) + k(a \cos\Omega t + b \sin\Omega t)$$

$$= (b\Omega + ka) \cos \Omega t + (bk - a\Omega) \sin\Omega t$$

$$= F_0 \cos\Omega t.$$

Equating coefficients of $\cos\Omega t$ and $\sin \Omega t$ on the two sides of this last equation leads to

$$b\Omega + ka = F_0 \qquad and \qquad bk - a\Omega = 0$$

and $$a = \frac{F_0 k}{k^2+\Omega^2}, \qquad b = \frac{F_0 \Omega}{k^2+\Omega^2} \qquad .$$

This agrees with the result obtained by variation of parameters.

In some situations, it is necessary to express the particular solution in a form that is consistent with the forcing term $f(t) = F_0 \cos\Omega t$. Note that

$$y_p(t) = \frac{F_0}{k^2+\Omega^2}[k \cos\Omega t + \Omega \sin\Omega t]$$

$$= \frac{F_0}{\sqrt{k^2+\Omega^2}}\left[\frac{k}{\sqrt{k^2+\Omega^2}}\cos\Omega t + \frac{\Omega}{\sqrt{k^2+\Omega^2}}\sin\Omega t\right].$$

Let

$$\alpha = \frac{k}{\sqrt{k^2+\Omega^2}} \quad \textit{and} \quad \beta = \frac{\Omega}{\sqrt{k^2+\Omega^2}}$$

and note further that

$$\alpha^2+\beta^2 = \frac{k^2}{k^2+\Omega^2} + \frac{\Omega^2}{k^2+\Omega^2} = 1.$$

Since the sum of α^2 *and* β^2 equals one, it follows that we can always find an angle θ such that $\alpha = \cos\theta,\ \textit{and}\ \beta = \sin\theta$;

i.e. $$\frac{\beta}{\alpha} = \frac{\cos\theta}{\sin\theta} = Tan\theta \quad \textit{or} \quad \theta = Tan^{-1}\left(\frac{\beta}{\alpha}\right).$$

Now we can write

$$y_p(t) = \frac{F_0}{\sqrt{k^2+\Omega^2}}\left[\frac{k}{\sqrt{k^2+\Omega^2}}\cos\Omega t + \frac{\Omega}{\sqrt{k^2+\Omega^2}}\sin\Omega t\right]$$

$$= \frac{F_0}{\sqrt{k^2+\Omega^2}}[\cos\theta\cos\Omega t + \sin\theta\sin\Omega t]$$

$$= \frac{F_0}{\sqrt{k^2+\Omega^2}}\cos(\Omega t - \theta),$$

from which it is evident that $y_p(t)$ is a periodic function having the same period as the forcing term, but with a phase shift equal to θ. In addition it is clear that the amplitudes of the forcing term and the particular solution are equal to F_0 *and* $\frac{F_0}{\sqrt{k^2+\Omega^2}}$ respectively.

.