

## Sample Final Exam

### Part A

1. State the completeness axiom. What is the point of this axiom; what is it designed to achieve?
2. The nested interval theorem and the Balzano-Weierstrass theorem can be stated as theorems about the real numbers. What do they each say ?
3. Define convergence for a sequence of real numbers  $\{x_n\}$ .
4. Define what it means for  $\{x_n\}$  to be a Cauchy sequence of real numbers.
5. State two equivalent definitions for what it means for  $f(x)$  to be continuous at the point  $x = c$ .
6. Give two consequences that follow if  $f(x)$  is continuous for  $x \in [a, b]$ .
7. Define what it means for  $f(x)$  to be differentiable at  $x = c$ . Give an example to illustrate an  $f$  and a point  $c$  where  $f'(c)$  fails to exist.
8. State two equivalent definitions for what it means for  $f(x)$  to be integrable on  $[a, b]$ .
9. Give three conditions on  $f$  sufficient to imply that  $f$  is integrable on  $I = [a, b]$
10. Suppose  $f(x)$  is the staircase function whose value at  $x$  is equal to the smallest integer that is larger than  $x$ . Can you use the fundamental theorem of calculus to evaluate  $\int_{-2}^2 f(x) dx$  ? Explain.
11. Under what conditions is it true that  $\frac{d}{dx} \int_a^x f = f(x)$  ?
12. Under what conditions is it true that  $\int_a^b f = F(b) - F(a)$  ?
13. Define a metric on  $X = C[a, b]$  so that it leads to a complete metric space.
14. What does it mean to say the metric space in the previous problem is complete?

### Part B

1. Explain what it means to say the rationals are dense in the reals. Which axiom defining the real numbers is crucial to this property?
2. Suppose  $A$  denotes a set of real numbers containing an infinite number of elements. If

$|x| \leq 5$  for all  $x$  in  $A$ , then what conclusions can you draw from this information?

3. Give two equivalent definitions for an open set. Give two equivalent definitions for a closed set. Can a set be open and closed? Can a set be neither open nor closed? Explain with examples.
4. Explain the difference between a boundary point and an isolated point of a set  $A$ ; between a boundary point and an interior point.; between an accumulation point and an isolated point
5. Can a set consist of only isolated points / interior points/ boundary points? Explain with examples
6. Suppose  $\{a_n\}$  is a sequence of positive numbers with  $a_{n+1} \leq a_n$ . Is this enough information to determine if  $\{a_n\}$  converges? Explain.
7. Can a bounded sequence have more than one accumulation point? Explain.
8. Can an unbounded sequence have more than one accumulation point? Explain.
9. Can a convergent sequence have more than one accumulation point? Explain.
- 10 Can a Cauchy sequence be unbounded? Explain.
11. Can a Cauchy sequence be divergent? Explain.
- 12.Can a Cauchy sequence have more than one accumulation point? Explain
13. Give an example of a sequence with three/ no accumulation points.
14. Can a sequence have no accumulation points? Can such a sequence converge?
15. Is every monotone sequence convergent? Is every convergent sequence monotone?
16. Does every bounded sequence have an accumulation point? Does every monotone sequence have an accumulation point?
17. Is every Cauchy sequence monotone? Is every monotone sequence Cauchy? Is every bounded monotone sequence Cauchy?
18. Suppose  $f(x)$  is continuous at  $x = c$ . Give the  $\varepsilon, \delta$  definition of what this means. Define what this means in terms of sequences. Define what this means in terms of function limits.
- 19 What does it mean to say  $f(x)$  has a finite jump discontinuity at  $x = c$ ? If  $f(x)$  has a jump discontinuity at  $x = 0$ , explain how you would use the sequence definition of continuity to prove  $f$  is not continuous at  $x = 0$ .
20. Give an example of a function that is continuous but not uniformly continuous on  $(-1, 1)$ .

Give an example of a function that is uniformly continuous on  $(-1, 1)$ . Explain the difference between continuity and uniform continuity.

21. What are the hypotheses and conclusions of: the extreme value theorem, the intermediate value theorem? Give an example where **one** of the hypotheses is not satisfied and the conclusion then fails to hold

22. State the persistence of sign result for continuous functions.

23. How are the properties of continuity, monotonicity and injectivity related?

24. Use the MVT for derivatives to prove that if  $f(x)$  is differentiable on  $(a, b)$  then  $f$  is continuous at each  $x$  in  $(a, b)$ .

25. What is the relation between Rolle's theorem and the MVT for derivatives.?

26. Explain why a positive value for the derivative of  $f(x)$  at a point implies  $f$  is increasing in a neighborhood of the point.

27. Explain why the derivative is zero at a local extreme point. Is a point where the derivative is zero necessarily a local extreme point?

28. Sketch the graph of a function which has an absolute maximum on  $[a, b]$  but the derivative is not zero at any point of  $[a, b]$ . There are two ways this can happen. Sketch the graph for both cases.

29. What are the conditions that must be satisfied in order for L'Hopital's rule to apply in evaluating a limit?

30. Suppose  $f$  is defined and bounded on  $I$  and  $P \in \Pi[I]$ . then  $\sigma[f, P] \leq \Sigma[f, P]$ . Explain how the sums  $\sigma[f, P^*]$  and  $\Sigma[f, P^*]$  are related to  $\sigma[f, P]$  and  $\Sigma[f, P]$  if  $P^*$  is a refinement of  $P$ . Explain why it is true that  $\sigma[f, P] \leq \Sigma[f, Q] \quad \forall P, Q \in \Pi[I]$

31. Suppose  $f(x) = 0$  for all  $x \in R$  except  $x = 0$  where  $f(0) = 100^{100^{100}}$ . Can you use the fundamental theorem of calculus to evaluate the integral  $\int_{-1}^1 f(x) dx$ ? Explain how to evaluate this integral.

32. Explain what is meant by an improper integral and define what it means for an improper integral to be convergent.

33. Suppose  $f$  is integrable on  $[a, b]$ . Use the integral to define the average value for  $f$  on  $[a, b]$ . If  $f$  is continuous on  $[a, b]$  show there exists a point  $c$  in  $[a, b]$  where  $f$  assumes this average value.

34. Use the fundamental theorem of calculus to find:  $\frac{d}{dx} \int_{x^2}^{e^x} f(s) ds$  if  $f = F'$

35. Find  $\lim_{x \rightarrow 0} \frac{1}{x} \int_{-3x}^{2x} e^{-t^2} dt$

36. Define a metric on  $X = C[a, b]$  so that it leads to a metric space that is not complete. What does "not complete" mean?

**Part C**

1. Let  $p$  be an accumulation point for a set  $A \subset \mathbb{R}$ . Show that there exists a sequence of  $A$  that converges to  $p$ . Does this imply that  $p$  belongs to  $A$  ?

2. Prove that  $a_n = \frac{1}{n^2}$  is a Cauchy sequence. Does this imply the sequence converges?

3. Use the  $\epsilon - \delta$  definition of continuity to show  $f(x) = x^2$  is continuous at  $x = c$

4. Tell whether the following statements are true or false

If the statement is false, give an example that shows it is false. }  
 If the statement is true, state a result that supports that conclusion. }

- a. If  $f$  is continuous on  $D$  and  $D$  is bounded then there exists  $M$  such that  $f(x) \leq M \forall x \in D$
- b. If  $f$  is continuous on  $dom f$  and  $dom f$  is bounded, then  $\{f(x_n)\}$  must be bounded for any  $\{x_n\} \subset dom f$ .

5. Suppose  $f$  is differentiable at every  $x$ . Show that  $g(x) = f(x + c)f(x - c)$  is differentiable at every  $x$ .

6. If  $\frac{df}{dx} \geq c > 0$  for all  $x$ , show that  $\lim_{x \rightarrow \infty} f(x) = \infty$ .

7. For  $f$  integrable on  $I = [a, b]$  let  $F(x) = \int_x^b f$

- a. show that  $F$  is uniformly continuous on  $I$
- b. Show that at each  $x$  in  $I$  where  $f$  is continuous, we have  $\lim_{h \rightarrow 0} D_h F(x) = f(x)$

8. Suppose  $f$  and  $g$  are continuous on  $I = [a, b]$  and that  $\int_I f = \int_I g$ . Prove there exists  $c$  in  $I$  such that  $f(c) = g(c)$

9. Define a metric on  $X = C[a, b]$  so that the resulting metric space is not complete. Explain how you would show the space is not complete.

10. Let  $X = C[a, b]$  denote the complete metric space defined in problem 13 of part A. Show that if  $G[x(t)] = \int_a^b x(t)dt$  for  $x(t) \in X$ , then is a continuous function on  $X$  with values in  $\mathbb{R}$ .