Problems Chapter 4    Differentiation

1. Use the definition to find the derivative of the following functions:
   a. \( x^4 \)
   b. \( \frac{1}{x^2} \)
   c. \( \frac{1}{\sqrt{x}} \)
   d. \( \cos x \)

2. Use the rules for derivatives to find the derivative of the following functions:
   a. \( \frac{x}{1 + x^2} \)
   b. \( \sqrt{x^2 - 3x + 1} \)
   c. \( \tan(x^2) \quad |x| < \pi/2 \)
   d. \( \cos^{-1}(x) \)

3. Which of the following functions is differentiable at \( x = 0 \)? Which of the following functions is differentiable for \( x \neq 0 \)? Where the derivative exists, is it continuous?
   a. \( x|x| \)
   b. \( |x + 1| + |x - 1| \)
   c. \( x \cos \left( \frac{1}{x} \right) \)
   d. \( x^2 \cos \left( \frac{1}{x} \right) \)
   e. \( f(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases} \)

4. Let \( f(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ ax & \text{if } x < 0 \end{cases} \)
   a. For which values of \( a \) is \( f \) continuous at \( x = 0 \)?
   b. For which values of \( a \) is \( f \) differentiable at \( x = 0 \)?
   c. When \( f \) is differentiable at \( x = 0 \), does \( f''(0) \) exist?

5. Find all the points where \( f(x) = \sqrt{1 - \cos x} \) is not differentiable. Explain why the derivative fails to exist.

6. Let \( H(x) = 1 \) for \( x > 0 \) and \( H(x) = 0 \) for \( x \leq 0 \).
   a. For what values of \( p \in \mathbb{R} \) is the function \( F(x) = x^p H(x) \) continuous at \( x = 0 \)?
   b. For what values of \( p \in \mathbb{R} \) is the function \( F(x) = x^p H(x) \) differentiable at \( x = 0 \)?
   c. Compute \( F'(x) \) for each \( p \) and at each \( x \) where the derivative exists.
7. Find a function \( f(x) \) such that
\[
f'(-1) = f'(0) = f'(1) = 0 \\
f''(-1) > 0 \quad f''(0) < 0 \quad f''(1) > 0
\]

8. Give an example of a function \( f(x) \) that is continuous on \([-1, 1]\) such that:
   a. \( f \) has a maximum at some \( c \in (-1, 1) \) but \( f'(c) \neq 0 \)
   b. \( f'(c) = 0 \) at some \( c \in (-1, 1) \) but \( c \) is neither the max nor the min for \( f \) on \([-1, 1]\).
   c. \( f' \) is zero at both the max and the min for \( f \) on \([-1, 1]\).

9. Suppose \( f \) is continuous and differentiable on \([-1, 1]\) and that \( f'(x) \) is continuous on \([-1, 1]\) as well. Show that \( f \) is Lipschitz continuous.

10. A function \( F(x) \) is said to be periodic with period \( L \) if \( F(x + L) = F(x) \) for all \( x \). Suppose \( F \) is periodic and continuous on \( \mathbb{R} \). Then show that \( F \) is bounded and uniformly continuous on \( \mathbb{R} \).

11. Using 580 feet of fence wire, build a rectangular pen of maximum area by making use of the fixed walls of length 200 feet and 400 feet, respectively as shown below. Note that there are 3 different configurations that make use of the fixed walls. In each of these configurations, the area is equal to \( x_j y_j \) but in each case, \( x_j \) and \( y_j \) satisfy a different condition (e.g. in the second case the condition is, \( x_2 + y_2 = 580 \)).
   a. In each of the three configurations: express \( y \) in terms of \( x \); what are the max and min values for \( x \) and \( y \) in each configuration? express the area in terms of \( x \)
   b. determine the maximum area that can be achieved and explain how the Max-min theorem must be used in finding the maximum area in this problem.
   c. plot the graph of \( A(x) \) versus \( x \) and show where the max occurs.
12. Find all real values of $a$ for which the function $f(x) = x^3 + ax^2 + 3x + 15$ is strictly increasing on $(0, 1)$.

13. Give an example of a function $f(x)$ which is differentiable with a differentiable derivative $f'(x)$ but whose second derivative $f''(x)$ is discontinuous.

14. Suppose $f$ is differentiable at every $x$. Prove $g(x) = f(x)^2$ is differentiable at every $x$.

15. Suppose $f$ is differentiable at every $x$. Prove $g(x) = f(x-1)f(x+1)$ is differentiable at every $x$.

16. Find all the critical points for $f(x) = x^4$.

17. Suppose $f \in C[a,b] \cap D(a,b)$ and $|f'(x)| \leq 2$ on $[a,b]$. Prove that $f$ is uniformly continuous on $[a,b]$. Give an example of a function that is uniformly continuous on $[a,b]$ but its derivative is not bounded on $[a,b]$.

18. Suppose $f \in C[a,b] \cap D(a,b)$ and $f$ has an absolute max at an interior point $c \in (a,b)$. Does this imply $f'(c) = 0$? If $f'(c) = 0$ at an interior point $c \in (a,b)$, does this imply that $f(x)$ has a max or a min at $x = c$?

19. If $x(t) = a \cos t$ and $y(t) = b \sin t$, find the extreme values for $\frac{dy}{dx}$, and $\frac{d^2y}{dx^2}$ and locate the points on the path where they occur.

20. Consider the function, $f(x) = 1 - (x - 1)^{2/3}$ on $[0, 2]$. Does Rolle’s theorem apply in this case? Explain.

21. Show that between any two zeroes of $e^x \sin x = 1$, there is at least one real zero of $e^x \cos x = -1$.

22. Show that if $0 < a < b$, then $(1 - a/b) < \ln(b/a) < b/a - 1$.

23. Prove that $\frac{\sin x}{x}$ is strictly decreasing on $(0, \pi/2)$. Then prove that $0 \leq \sin x \leq 2x/\pi$ for $0 \leq x \leq \pi/2$.

24. If $a \beta > 0$ then show that $x^3 + ax^2 + \beta = 0$ has one real root.

25. If $f'(x) > c > 0$ for all $x \geq 0$, show that $\lim_{x \to \infty} f(x) = \infty$.

26. If $f \in C^2(R)$ and $f(x) = 0$ has 3 real roots, show that there is some $z \in R$ where $f''(z) = 0$.

27. Suppose $f \in C[a,b] \cap D(a,b)$ Let $S$ denote the set of slopes of all possible secant lines for $y = f(x)$ for $x \in (a,b)$ and let $D$ denote the set of all possible values for $f'(x)$ for $x \in (a,b)$. Show that $S \subset D$ but $D$ need not equal $S$. 

3. 
28. Show that the conclusions of the mean value theorem may fail if we drop the condition that \( f \) is differentiable at every point of \((a, b)\). What if we allow \( f \) to be discontinuous at \( a \) or \( b \)?

29. Show that if \( f \in C[a, b] \) and \( f \) has first and second derivatives at each point of \((a, b)\), then there exists a \( c \in (a, b) \) such that
\[
 f(b) = f(a) + f'(a)(b - a) + \frac{1}{2}f''(c)(b - a)^2.
\]

30. If \( f\left(\frac{x+y}{2}\right) = \frac{f(x)-f(y)}{x-y} \) for \( x \neq y \), what can you say about \( f(x) \)?

31. Use what you know about geometric series and Taylor series to obtain the result,
\[
\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \xi^6 \quad \text{for} \quad 0 < \xi < x.
\]

32. Use the result of the previous problem to obtain,
\[
\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7} \quad \text{for} \quad 0 < \xi < x.
\]

33. Use the result of problem 31 to obtain
\[
\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - 6x^5 + 7x^6 \quad \text{for} \quad 0 < \xi < x.
\]

34. We say \( f(x) = O(x^p) \) as \( x \to 0 \), if \( f(x)x^{-p} \) tends to a finite limit as \( x \to 0 \). In this case we say \( f(x) \) vanishes to order \( p \) at \( x = 0 \). Find the order for \( f(x) = \sin^2 2x \) and \( g(x) = 1 - \cos 3x \).

35. Find \( O(x^p) \) for \( f(x) = x^3 - \sin^3 x \) and \( g(x) = x - \ln(1+x) - 1 + \cos x \).

36. Find real numbers \( A \) and \( a \) such that
   \begin{enumerate}
   \item \( \lim_{x \to 0} \frac{1 - \cos x}{Ax^a} = 1 \)
   \item \( \lim_{x \to 0} \frac{\sin x - x}{Ax^a} = 1 \)
   \end{enumerate}

37. Can we use L’Hopital’s rule be used to determine the limit of the sequence, \( a_n = n^2e^{-n^2} \).

38. Can L’Hopital’s rule be used to find
\[
\lim_{x \to 0} \frac{x^2 \sin(1/x)}{\sin x}
\]

39. Suppose \( f \) is such that for some \( C > 0 \), \( |f(x) - f(y)| \leq C|x - y|^2 \). Prove that \( f \) is a constant function.

40. Attack or defend the statement “if \( f \) is differentiable at all \( x \) and \( f'(a) = 0 \) at some point \( x = a \), then \( f \) is not injective.”
41. Show \( f(x) = x^3 + x^2 + 8 \) has an inverse and find the derivative of \( f^{-1} \).

42. Suppose \( f, g \in D(R) \) and \( f(0) = g(0) \). Show that if \( f' \leq g' \) on \( R \) then \( f(x) \leq g(x) \) for all \( x \geq 0 \).

43. Suppose \( f \in D(R) \) and \( f(0) = 0 \). Show that if \( 1 \leq f'(x) \leq 2 \) on \( R \) then \( x \leq f(x) \leq 2x \) for all \( x \geq 0 \).

44. Suppose \( f \in D(R) \) and \( |f'(x)| < 1 \) \( \forall x \) For \( s_0 \in R \) define \( s_n = f(s_{n-1}) \) for \( n = 1, 2, \ldots \)
Show that \( \{s_n\} \) is a Cauchy sequence.

45. Evaluate the following limit:
\[
\lim_{x \to 0} \left[ \frac{1}{\sin x} - \frac{1}{x} \right]
\]

46. Evaluate the following limit:
\[
\lim_{x \to 0} \frac{1}{x^2}
\]

47. Evaluate the following limit:
\[
\lim_{x \to 0} \left[ \frac{e^{2x} - \cos x}{\sin x} \right]
\]

48. Evaluate the following limit:
\[
\lim_{x \to 0} \left[ x + e^x \right] \frac{1}{x}
\]

49. Evaluate the following limit:
\[
\lim_{x \to \infty} \left[ 1 + \frac{1}{x} \right]^{2x}
\]

50. Show that if \( f(x) = \begin{cases} \frac{e^{-1/x}}{x} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases} \) then \( f^{(n)}(0) = 0 \) for \( n = 1, 2, \ldots \).