Problems for Chapter 3.

- Let \( A \) denote a nonempty set of reals. The complement of \( A \), denoted by \( A^c \), or \( A^C \) is the set of all points \( x \) not in \( A \).
- We say that \( x \) belongs to the interior of \( A \), \( x \in \text{Int}(A) \), if there exists a positive \( \varepsilon \) such that \( N_\varepsilon(x) \subset A \).
- We say that \( x \) belongs to the boundary of \( A \), \( x \in \partial A \), if for every positive \( \varepsilon \) the neighborhood \( N_\varepsilon(x) \) contains points of \( A \) and points of \( A^c \).
- We say that \( x \) is an isolated point of \( A \) if there exists a positive \( \varepsilon \) such that \( N_\varepsilon(x) \) contains no points of \( A \) other than \( x \).
- A set \( A \) is said to be open if all the points of \( A \) are interior points. A set \( A \) is said to be closed if \( A^c \) is open.

1. Find all the interior points, isolated points, accumulation points and boundary points for
   a. \( \mathbb{N}, \mathbb{Q}, \text{ and } \mathbb{R} \)
   b. \((a, b) \text{ and } [a, b]\)
   c. \(\mathbb{R} \text{ with } \mathbb{N} \text{ removed} \)
   d. \(\mathbb{R} \text{ with } \mathbb{Q} \text{ removed} \)

2. Give an example of:
   a. A set with no accumulation points.
   b. A set with infinitely many accumulation points, none of which belong to the set.
   c. A set that contains some, but not all, of its accumulation points

3. Give an example of a set with the following properties or explain why no such set can exist:
   a. A set with no accumulation points and no isolated points
   b. A set with no interior points and no isolated points
   c. A set with no boundary points and no isolated points

4. Is every interior point of \( A \) an accumulation point? Is every accumulation point of \( A \) an interior point?

5. Let \( x \) be an interior point of \( A \) and suppose \( \{x_n\} \) is a sequence of points, not necessarily in \( A \), but converging to \( x \). Show that there exists an integer \( N \) such that \( x_n \in A \ \forall \ n > N \)

6. Prove the following statements
   a. if \( G_n \) is open for every \( n \in \mathbb{N} \), then \( \bigcup_{n \in \mathbb{N}} G_n \) is open
   b. \( F \) is closed if and only if \( F \) contains all its boundary points
7. Find the interior and boundary for each of the following sets.
   \[ a. \quad A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \]
   \[ b. \quad A = \{ x \in \mathbb{Q} : 0 < x^2 < 2 \} \]

8. Use both the definition of limit and a sequence approach to establish
   \[ \lim_{x \to 2} \frac{1}{1 - x} = -1 \]

9. Use both the definition of limit and a sequence approach to establish
   \[ \lim_{x \to 0} \frac{x^2}{|x|} = 0 \]

10. Use both the definition of limit and a sequence approach to establish
    \[ \lim_{x \to 1} \frac{x}{1 + x} = \frac{1}{2} \]

11. Show that the limit: \( \lim_{x \to 0} \frac{x}{|x|} \) does not exist

12. Show that the limit: \( \lim_{x \to 0} \sin\left(\frac{1}{x^2}\right) \) does not exist

13. Show that the limit: \( \lim_{x \to 0} \cos\left(\frac{1}{x}\right) \) does not exist

14. Show that the limit: \( \lim_{x \to 0} \frac{1}{\sqrt{x}} \) does not exist

15. Prove that if \( a_n \geq 0 \ \forall n \) and \( a_n \to A \), then \( \sqrt{a_n} \to \sqrt{A} \)

16. Find \( \lim_{x \to 0} \frac{(x + 1)^2 - 1}{x} \) or show the limit does not exist

17. Find \( \lim_{x \to 1} \frac{\sqrt{x} - 1}{x - 1} \) or show the limit does not exist

18. Find \( \lim_{x \to 0} \frac{\sqrt{1 + 2x} - \sqrt{1 + 3x}}{x + 2x^2} \) or show the limit does not exist

19. Find \( \lim_{x \to 0} x \sin\left(\frac{1}{x}\right) \) or show the limit does not exist

20. Find \( \lim_{x \to 0} \sqrt{x} \sin\left(\frac{1}{x}\right) \) or show the limit does not exist

21. Find \( \lim_{x \to 0} x^2 \cos\left(\frac{1}{x}\right) \) or show the limit does not exist

22. Find \( \lim_{x \to 0} x \cos\left(\frac{1}{x^2}\right) \) or show the limit does not exist

23. Given that \( x - \frac{1}{6}x^3 \leq \sin x \leq x \) for \( x \geq 0 \), find \( \lim_{x \to 0} \left( \frac{\sin x}{x} \right) \)

24. Given that \( x - \frac{1}{6}x^3 \leq \sin x \leq x \) for \( x \geq 0 \), find \( \lim_{x \to 0} \left( \frac{\sin x}{\sqrt{x}} \right) \)

25. Given that \( 1 - \frac{1}{2}x^2 \leq \cos x \leq 1 \) for \( x \geq 0 \), find \( \lim_{x \to 0} \left( \frac{\cos x - 1}{x} \right) \)
26. Given that \( 1 - \frac{1}{2} x^2 \leq \cos x \leq 1 \) for \( x \geq 0 \), find \( \lim_{x \to 0} \left( \frac{\cos x - 1}{\sqrt{x}} \right) \).

27. Prove that \( f(x) = |x| \) is continuous at all values of \( x \). Does \( x = 0 \) require special attention?

28. Let \( f(x) = \frac{\sin x}{\sqrt{x}} \). Can \( f(0) \) be defined in such a way that \( f \) is continuous for all \( x \)?

29. For each of the following functions, find the points where they are discontinuous or give a reason they are continuous for all \( x \):
   
   a. \( \frac{x^3 + 1}{x^2 + 1} \)
   
   b. \( \sin^2 x \cos x \)
   
   c. \( \frac{\cos x - 1}{\sqrt{x}} \)
   
   d. \( \frac{x - 2}{|x - 2|} \)

30. The functions \( f(x) = \frac{x^3 - 3x^2 + 2}{x^2 - 1} \) and \( g(x) = \frac{x^2 + 4x - 5}{x^3 - 2x^2 + x} \) are each undefined at two points. Are these singularities removable?

31. If \( F(x) = f(x) - g(x) + h(x) \), \( G(x) = g(x) + 2h(x) \) and \( H(x) = 2g(x) - h(x) \) are all continuous on \( \mathbb{R} \), then is it the case that \( f(x) \), \( g(x) \), and \( h(x) \) are also all continuous on \( \mathbb{R} \).

32. Give an example of functions \( f \) and \( g \) that are both discontinuous at \( x = c \) but (i) \( f + g \) is continuous at \( x = c \) (ii) \( fg \) is continuous at \( x = c \).

33. Suppose \( f \) is continuous for all \( x \) and that \( f(x) = 0 \) for every rational \( x \). Show that \( f(x) = 0 \) for all \( x \).

34. If \( f \) and \( g \) are continuous on \( \mathbb{R} \) and \( f\left( \frac{p}{q} \right) = g\left( \frac{p}{q} \right) \) for all non-zero integers, \( p, q \), then is it true that \( f(x) = g(x) \) for all \( x \in \mathbb{R} \)?

35. Suppose \( f(x) = \begin{cases} 
2x & \text{if } x = \text{rational} \\
x + 3 & \text{if } x = \text{irrational} 
\end{cases} \). Then find all the points where \( f \) is continuous.

36. Suppose \( f \) is defined on \( (0, 1) \) and that \( |f(x)| \leq 1 \) for all \( x \in (0, 1) \). If \( \lim_{x \to 0} f(x) \) does not exist then show there must be sequences \( a_n, b_n \) converging to 0 for which the sequences \( f(a_n) \) and \( f(b_n) \) converge but to different limits.

37. Suppose \( f \) is continuous at all \( x \) and let \( P = \{ x : f(x) > 0 \} \). If \( c \in P \) then show that there is an \( \varepsilon > 0 \) such that \( N_\varepsilon(c) \subseteq P \).

38. Suppose \( f \) and \( g \) are continuous for all \( x \) and let \( S = \{ x : f(x) \geq g(x) \} \). If \( c \) is an accumulation point for \( S \), show that \( c \in S \).

39. If \( f \) and \( g \) are continuous on \([0, 1]\) and \( f(x) > g(x) \) for \( 0 \leq x \leq 1 \) then does there exist a \( p > 0 \) such that \( f(x) \geq g(x) + p \) for \( 0 \leq x \leq 1 \).

40. If \( f \) and \( g \) are continuous on \([0, 1]\) and \( f(x) > g(x) \) for \( 0 < x < 1 \) then there does exist a \( p > 0 \) such that \( f(x) \geq g(x) + p \) for \( 0 \leq x \leq 1 \).
41. Let \( f(x) = \begin{cases} x + 1 & \text{if } 0 \leq x \leq 1, \ x \neq \frac{1}{2} \\ 0 & \text{if } x = \frac{1}{2} \end{cases} \) and \( g(x) = \begin{cases} \frac{1}{x} & \text{if } 0 < x < 1 \\ 0 & \text{if } x = 0 \end{cases} \)

Explain how you would prove the continuity or lack of continuity for these two functions. i.e., in each case, cite a theorem which supports your answer.

42. Let \( f(x) = \frac{x}{x^2 + 1} \) and let \( A = f\{1 \leq x \leq 8\} \subseteq \text{rng}[f] \) \( M = f^{-1}\{.2 \leq y < .4\} \subseteq \text{dom}[f] \). In answering the following questions, give reasons (i.e., cite theorems or give examples) for your answers.

   a. Is \( A \) closed? Is it bounded?
   b. Find the sup and the inf for \( A \). Do these belong to the set?
   c. Find the set of all \( y \) that belong to \( A \)
   d. Find the set of all \( x \) that belong to \( M \)
   e. Either prove that \( M \) is closed or show that it is not closed.

43. Suppose \( A \) is an infinite subset of the reals and that \( p \) is the LUB for \( A \) but \( p \) does not belong to \( A \). Tell whether the following statements are true or false and give coherent reasons for your answer.

   a. \( p \) is an accumulation point for \( A \)
   b. There is a “largest value” \( x \) in \( A \) such that \( x \leq p \).

44. State the compact range (extreme value, intermediate value) theorem. Give an example where one of the hypotheses is not satisfied and the conclusion then fails to hold.

45. State the persistence of sign theorem. Explain the use of this theorem to prove the following result: If \( f \) and \( g \) are continuous on \( \mathbb{R} \) and \( f(x) = g(x) \) for each rational \( x \), then \( f(x) = g(x) \) for all real \( x \).

46.

   a. State a condition on \( f(x) \), and \( D \) that implies that \( f \) is uniformly continuous on \( D \)
   b. State a condition on \( f(x) \), that implies that \( f \) is uniformly continuous on \( (a, b) \)
   c. Is \( f(x) = \frac{1}{1 + x^2} \) uniformly continuous on \( (0, \infty) \)?

47. Use the intermediate value theorem to tell how many real zeroes exist for the function \( f(x) = \sin x - \cos x \) as well as to determine the approximate location of these zeroes.
48. Suppose $f$ and $g$ are continuous on $(-\infty, \infty)$ and that $f(x) = g(x)$ at every rational value for $x$. Use the persistence of sign result to show that $f$ and $g$ must be equal at every real value.

49. Given that the function $f(x) = \sin\left(\frac{1}{x}\right)$ is continuous on $(0, 1)$, is $f$ uniformly continuous on $(0, 1)$? Given that the function $g(x) = \sqrt{x}$ is continuous on $(0, 1)$, is $g$ uniformly continuous on $(0, 1)$?

50. Let $A$ denote the set of all the rational numbers between 0 and 1.
   a. Is this a closed set?
   b. Is it an open set?
   c. What are the accumulation points (boundary points, interior points) for $A$?

51. For each of the following statements, first, tell whether the following are true or false, then state a theorem which shows a statement is true or give a counter example that shows it is false.
   a. If $f(x)$ is continuous and injective on $[a, b]$ and $f(a) < f(b)$, then for all $x, y \in [a, b]$, $x < y$ if and only if $f(x) < f(y)$.
   b. There exist sequences $\{a_n\}$ which are bounded but which contain no convergent subsequences.
   c. For every real value, $x$, there is a sequence of rational numbers that converges to $x$.
   d. If $\{a_n\}$ in $D$ converges to $c \in D$, and $\{f(a_n)\}$ converges to $f(c)$, then $f$ must be continuous at $x = c$.
   e. If $F$ is continuous on $D$ where $F(x) = f(x)g(x)$, then $f$ and $g$ are continuous on $D$.

52. Use the intermediate value theorem to tell how many real zeroes exist for the function $f(x) = \sin x - \cos x$ as well as to determine the approximate location of these zeroes.

53. Use the intermediate value theorem to tell how many positive zeroes exist for the function $f(x) = \sin x - \frac{1}{x}$ as well as to determine the approximate location of these zeroes.

54. Tell whether the following statements are true or false and cite a theorem to justify your answer:
   a. If $f(x)$ is continuous and monotone on $[a, b]$ then $\forall x, y \in [a, b]$, $x < y$ implies $f(x) \neq f(y)$.
   b. If $\{a_n\}$ is a monotone sequence that does not converge then $\left\{\frac{1}{a_n}\right\}$ must tend to zero as $n$ tends to infinity.
c. If \( f(x) \) is continuous on \((-\infty, \infty)\) and \( f(x) = 0 \) if \( x \) is rational, then \( f(x) = 0 \) at every real \( x \).

d. If \( A \) is an infinite subset of the reals and \( p = \sup A \) but \( p \) is not in \( A \) then there is no largest \( x \) in \( A \) such that \( x \leq p \).

e. If \( F \) is continuous on \( D \) where \( F(x) = 5f(x) + 4g(x) \), then \( f \) and \( g \) are continuous on \( D \).

55. Let \( f(x) = \frac{\sin x}{\sqrt{x}} \). Can \( f(0) \) be defined in such a way that \( f \) is uniformly continuous on \([0, 1]\)? Is \( f \) uniformly continuous on \([0, \infty)\)?