Additional Problems  Sequences

1. Consider the sequence of prime numbers, 1, 2, 3, 5, 7, 11, . . . Is this really a sequence? How do you define \( a_n \)?

2. What is the next term in the sequence 3, 1, 5, 1, 7, . . . Give a definition for \( a_n \).

3. Find an \( N \) such that \( |a_n - L| \leq 10^{-3} \) for \( n > N \)
   a. \( a_n = \frac{2}{\sqrt{n} + 1} \)
   b. \( a_n = 1 - \frac{1}{n^3} \)
   c. \( a_n = 2 + 2^{-n} \)

4. Prove convergence/divergence for \( a_n = \frac{2n^2 + 5n - 6}{n^3} \)

5. Prove convergence/divergence for \( a_n = \frac{3n + 5}{6n + 1} \).

6. Prove convergence/divergence for \( a_n = \frac{n\sqrt{n} + 2 + 1}{n^2 + 4} \).

7. Prove convergence/divergence for \( a_n = \sqrt{n + 1} - \sqrt{n} \).

8. Prove convergence/divergence for \( a_n = \sqrt{n} (\sqrt{n + 1} - \sqrt{n}) \).

9. Suppose \( a_n \) assumes only integer values. Under what conditions does this sequence converge?

10. Show that the sequences \( a_n \) and \( b_n = a_{n+10^6} \) either both converge or both diverge.

11. Let \( s_1 = 1 \) and \( s_{n+1} = \sqrt{s_n + 1} \). List the first few terms of this sequence. Prove that the sequence converges to \((1 + \sqrt{5})/2\).

12. A subsequence \( \{a_{n_k}\} \) is obtained from a sequence \( \{a_n\} \) by deleting some of the terms \( a_n \), and retaining the others in their original order. Explain why this implies that \( n_k \geq k \) for every \( k \).

13. Which statements are true? Explain your answer.
   a. If \( \{a_n\} \) is unbounded then either \( \lim_{n \to \infty} a_n = \infty \) or else \( \lim_{n \to \infty} a_n = -\infty \)
   b. If \( \{a_n\} \) is unbounded then \( \lim_{n \to \infty} |a_n| = \infty \)
   c. If \( \{a_n\} \) and \( \{b_n\} \) are both bounded then so is \( \{a_n + b_n\} \)
   d. If \( \{a_n\} \) and \( \{b_n\} \) are both unbounded then so is \( \{a_n + b_n\} \)
   e. If \( \{a_n\} \) and \( \{b_n\} \) are both bounded then so is \( \{a_n b_n\} \)
   f. If \( \{a_n\} \) and \( \{b_n\} \) are both unbounded then so is \( \{a_n b_n\} \)

14. Which statements are true? Explain your answer.
   a. If \( \{a_n\} \) and \( \{b_n\} \) are both divergent then so is \( \{a_n + b_n\} \)
   b. If \( \{a_n\} \) and \( \{b_n\} \) are both divergent then so is \( \{a_n b_n\} \)
   c. If \( \{a_n\} \) and \( \{a_n + b_n\} \) are both convergent then so is \( \{b_n\} \)
   d. If \( \{a_n\} \) and \( \{a_n b_n\} \) are both convergent then so is \( \{b_n\} \)
   e. If \( \{a_n\} \) is convergent then so is \( \{a_n^2\} \)
If \( \{a_n\} \) is convergent then so is \( \{1/a_n\} \).

If \( \{a_n^2\} \) is convergent then so is \( \{a_n\} \).

15. Either give an example of a sequence with the following property or else state a theorem that shows why no such example is possible.
   a. a sequence that is monotone increasing but is not bounded
   b. a sequence that converges to 6 but contains infinitely many terms that are not equal to 6 as well as infinitely many terms that are equal to 6
   c. an increasing sequence that is bounded but is not convergent
   d. a sequence that converges to 6 but no term of the sequence actually equals 6.
   e. a convergent sequence with all negative terms whose limit is not negative
   f. an unbounded increasing sequence containing a convergent subsequence
   g. a convergent sequence whose terms are all irrational but whose limit is rational.

16. How are the notions of accumulation point of a set and limit point of a sequence related? How does this relate to the two formulations of the Bolzano-Weierstrass theorem?

17. Prove: If the Cauchy sequence \( \{a_n\} \) contains a subsequence \( \{a_{n_k}\} \) which converges to limit \( L \), then the original sequence must also converge to \( L \).

18. Show that \( 1 + a + a^2 + \cdots + a^n = \frac{1-a^{n+1}}{1-a} \) for \( a \neq 1 \) and any positive integer \( n \).
   Find \( \lim_{n \to \infty} (1 + a + a^2 + \cdots + a^n) \) for \( |a| < 1 \). What is the limit if \( |a| \geq 1 \)?

19. Let \( \{s_n\} \) be such that \( |s_{n+1} - s_n| \leq 2^{-n} \) for all \( n \in \mathbb{N} \). Prove that this is a Cauchy sequence. Is this result true under the condition \( |s_{n+1} - s_n| \leq \frac{1}{n} \)?

20. Let \( s_1 = 1 \) and \( s_{n+1} = \frac{1}{3}(s_n + 1) \) for \( n \geq 1 \). Find the first few terms of this sequence. Use induction to show that \( s_n > \frac{1}{2} \) for all \( n \). Show that this sequence is nonincreasing. Prove that the sequence converges and find its limit.

21. Let \( s_1 = 1 \) and \( s_{n+1} = \left( 1 - \frac{1}{4n^2} \right)s_n \) for \( n \geq 1 \). Determine if the sequence converges and, if it does, find the limit.

22. For each of the following sequences state a theorem which establishes the convergence/divergence:
   a. \( a_n = n^{1/3} \)
   b. \( a_n = \frac{n^2 + 3}{n + 2} \)
   c. \( a_n = (2 + 10^{-n})(1 + (-1)^n) \)
d. \( a_n = \frac{1}{n^2 + 3n + 2} \)

e. \( a_n = 1 + 2^{-n} \)

f. \( a_n = \sqrt{n + 1} \)

g. \( a_n = \sum_{k=1}^{n} \frac{1}{k} \) (hint: show that \( a_{2n} - a_n \) does not tend to 0 as \( n \to \infty \))

h. \( \{a_n\} = \{1, \frac{1}{2}, 1, \frac{1}{4}, 1, \frac{1}{8}, \ldots\} \)

23. Let 
\( a_1 = 0.1, a_2 = 0.101, a_3 = 0.101001, a_4 = 0.1010010001, a_5 = 0.101001000100001, \ldots \)
Show that this is a sequence of rational numbers that converges to a limit \( L \). Is the limit \( L \) rational?

24. Which statements are true?:
   a. a sequence is convergent if and only if all its subsequences are convergent.
   b. a sequence is bounded if and only if all its subsequences are bounded.
   c. a sequence is monotone if and only if all its subsequences are monotone.
   d. a sequence is divergent if and only if all its subsequences are divergent.

25. The sequence \( \{a_n\} \) has the property, \( \forall \varepsilon > 0, \exists N_\varepsilon \) such that \( |a_{n+1} - a_n| < \varepsilon \) when \( n > N_\varepsilon \). Is the sequence necessarily a Cauchy sequence?