1. Define:
   a) the **Least Upper Bound** of a nonempty set \( A \subset R \).

   \[ a = \text{Lub}(A) \iff \begin{align*}
   & \text{(i) } a = \text{UB}(A) \text{ and } a \leq \beta, \forall \beta = \text{UB}(A) \\
   & \text{(ii) } a = \text{UB}(A) \text{ and } \forall \varepsilon > 0 \ a - \varepsilon \neq \text{UB}(A)
   \end{align*} \]

   b) an **accumulation point** of a nonempty set \( A \subset R \)

   \[ p = \text{acc pt of } A \iff \forall \varepsilon > 0, \ \hat{N}(p) \cap A \neq \emptyset \]

   c) an **isolated point** of a nonempty set \( A \subset R \)

   \[ p = \text{isol pt of } A \iff \exists \varepsilon > 0, \ \hat{N}(p) \cap A = \emptyset \]

2. Prove ONE of the following:
   a) If \( p \) is the least upper bound for a nonempty set \( A \subset R \), but \( p \) does not belong to \( A \), then \( p \) must be an accumulation point for \( A \).

   If \( \alpha = \text{Lub}(A) \) then \( \forall \varepsilon > 0, \ \alpha - \varepsilon \neq \text{UB}(A) \)

   Then there exists \( x \in A \) such that \( \alpha - \varepsilon < x \leq \alpha \).

   But \( \alpha \notin A \), so \( x \neq \alpha \).

   That means \( \alpha - \varepsilon < x < \alpha \) so \( \forall \varepsilon > 0, \ \hat{N}(a) \cap A \neq \emptyset \)

   b) If \( A \) is a nonempty open set in \( R \), then \( \sup(A) \) does not belong to \( A \).

   If \( A \) is open then all pts in \( A \) are interior pts.

   If \( \sup(A) \in A \) then since \( \forall p \in A, \ \exists \varepsilon > 0, N_{\varepsilon}(p) \subset A \), i.e., \((p - \varepsilon, p + \varepsilon) \subset A \).

   But if \( p + \varepsilon \in A \) then \( p \neq \sup(A) \), so \( p = \sup(A) \) cannot belong to \( A \)

   c) If \( p \) is an accumulation point for \( A \) then \( A \) contains a sequence \( \{a_n\} \) converging to \( p \).

   If \( \text{acc pt of } A \) then \( \forall n, \ \hat{N}_{1/n}(p) \cap A \neq \emptyset \)

   For each \( n \), choose \( a_n \in \hat{N}_{1/n}(p) \).

   Then \( \forall n, \ |a_n - p| < \frac{1}{n} \) if \( n > \frac{1}{n} \) so \( a_n \rightarrow p \)

3. a) For the sequence \( a_n = \frac{4n^3 + n^2 + 2}{n^3 + n^2 + 2} \), choose an \( L \) and find an \( N \) such that \( |a_n - L| < 10^{-5} \) for \( n > N \).

   For large \( n \), \( a_n \approx 4 \) so
\[ |a_n - 4| = \left| \frac{4n^3 + n^2 + 2}{n^3 + n^2 + 2} - 4 \right| \]
\[ = \left| \frac{4n^3 + n^2 + 2}{n^3 + n^2 + 2} - \frac{4(n^3 + n^2 + 2)}{n^3 + n^2 + 2} \right| \]
\[ = \left| \frac{-3n^2 - 6}{n^3 + n^2 + 2} \right| \leq \frac{3}{n} < 10^{-5} \text{ if } n > 3 \cdot 10^5 \]

b) For \( \lim_{x \to 2} \frac{x^2}{x^2 + 1} \), choose an \( L \) and a \( \delta > 0 \) such that \( \left| \frac{x^2}{x^2 + 1} - L \right| < 10^{-5} \) for \( |x - 2| < \delta \)

For \( x \approx 2, \frac{x^2}{x^2 + 1} \approx 4/5 \) so
\[ \left| \frac{x^2}{x^2 + 1} - \frac{4}{5} \right| = \left| \frac{5x^2 - 4(x^2 + 1)}{5(x^2 + 1)} \right| \]
\[ = \left| \frac{x^2 - 4}{5(x^2 + 1)} \right| < \frac{x + 2}{5} |x - 2| \]

If \( |x - 2| < 1 \), then \( \frac{x^2}{10} \leq \frac{1}{10} \), so
\[ \left| \frac{x^2}{x^2 + 1} - \frac{4}{5} \right| < \frac{1}{10} |x - 2| < 10^{-5} \text{ if } |x - 2| < 10^{-4} \]

4. Is the sequence \( a_n = \frac{(-1)^n n^2}{n^2 + 1} \) bounded?
\[ |a_n| = \frac{n^2}{n^2 + 1} < 1, \text{ so it is a bounded sequence.} \]

Does the sequence contain a convergent subsequence?

The B-W theorem asserts that every bounded sequence contains a cvg subsequence

Does the sequence \( \{a_n\} \) converge?

Clearly \( a_{2n} \) converges to 1 while \( a_{2n+1} \) converges to -1, so by the uniqueness of limits theorem,
\( \{a_n\} \) is not convergent.

State theorems which justify your answers.

5. Find the following function limits and state a theorem that proves the limit or one that asserts the limit fails to exist:

a) \( \lim_{x \to \infty} \sin x \)

Let \( a_n = 2n\pi \) and \( b_n = 2n\pi + \pi/2 \). Then \( \sin(a_n) = 0 \) and \( \sin(b_n) = 1, \forall n. \)

Then by the uniqueness of limits theorem, the limit fails to exist.
b) \[ \lim_{x \to 0} \frac{e^{-x}}{e^x + e^{-x}} = \lim_{x \to 0} \frac{e^{-2x}}{1 + e^{-2x}} = \frac{\lim_{x \to 0} e^{-2x}}{\lim_{x \to 0}(1 + e^{-2x})} = \frac{0}{1} \]

by the arithmetic with limits theorem

alternatively, \(0 < \frac{e^{-x}}{e^x + e^{-x}} < e^{-x}\) so the limit is 0 by the squeeze theorem

c) \[ \lim_{x \to 0} x^2 \cos x \]

\(-x^2 \leq x^2 \cos x \leq x^2\) so the limit is 0 by the squeeze theorem

alternatively,

\[ \lim_{x \to 0} x^2 \cos x = \lim_{x \to 0} x^2 \lim_{x \to 0} x^2 \cos x = 0 \cdot 1 \text{ by the arithmetic with limits theorem} \]