Lecture Notes, M261-004, Tangent Planes and Differentials

Sept 30, 2008

We have already talked about the idea of using the gradient of a function to approximate it. We consider this topic in more detail today. I will take a slightly different approach than the book that will hopefully complement it. We look at the following topics:

- The Idea of Linearization
- Tangent Planes and Normal Lines
- Differentials and Estimating Change in A Specific Direction
- Error Analysis

1 The Idea of Linearization

A linear function, roughly speaking, is a function where we multiply the input by some constant in an appropriate way and then add something to get the output. For a real-valued function of one variable, for example,

$$f(x) = ax + b$$

is the standard form of a linear function. For vector-valued function of one variable, a linear function looks like

$$\mathbf{r}(t) = \mathbf{a}t + \mathbf{b}$$

A linear vector-valued function of several variables would look like

$$\mathbf{r}(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$$

Where A is a matrix, $\mathbf{b}$ is a vector, and $\mathbf{x}$ is the input. A linear real-valued function of several variables looks like

$$f(\mathbf{x}) = \mathbf{a} \cdot \mathbf{x} + b$$

where $\mathbf{a}$ is a vector and b is a scalar.

We like linear functions because they are easy to evaluate. This is one very important reason why we care about derivatives. We often have a function that is hard to evaluate – maybe even impossible to evaluate precisely. The way we solve this problem is by replacing the function that is hard to evaluate with a linear function that is close to the original in some way. Now, the linear function is easy to evaluate and we can do things to make sure that it is close enough to the original function for our purposes, (mostly beyond the scope of this course.) The key idea is replacing a function with a linear function that is close to it. We use derivatives to find these linear functions, and the linear function is called a **linearization** of the function.

2 Tangent Planes and Normal Lines

In the case of a real-valued function of two variables or a two-dimensional surface in three-dimensional space, the linearization takes the form of a tangent plane. We consider the level surface of a function of three variables – that is, the set of points defined by

$$f(x, y, z) = c$$

Note that this also covers functions of two variables since in that case we change

$$z = f(x, y)$$

into

$$f(x, y) - z = 0$$

We learned last time that the level curve of a function of two variables is perpendicular to the gradient vector. This is also the case for the level surface of a function of three variables, using the same argument. (Repeating the argument in three variables is a good exercise.) This means that the tangent plane has the gradient of f as its normal vector, and will take the form

$$f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0) = 0$$

This is the tangent plane for the level surface $f(x, y, z) = c$ at the point (x_0, y_0, z_0). Remember that this could also be thought of as

$$\nabla f \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

If z was given as a function of x and y, this reduces to

$$f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) - (z - z_0) = 0$$

since we are now looking at

$$f(x, y) - z = 0$$

We can also find the normal line for the level surface at this point. It is parallel to the normal vector of the tangent plane, or the gradient. Hence we have

$$\begin{aligned} x &= x_0 + f_x(P_0)t \\ y &= y_0 + f_y(P_0)t \\ z &= z_0 + f_z(P_0)t \end{aligned}$$

or

$$\mathbf{r}(t) = t\nabla f + \langle x_0, y_0, z_0 \rangle$$

The tangent plane is the linearization of the function/surface near the point.

3 Differentials and Estimating Change in A Specific Direction

If we have our tangent plane approximation to a function of two variables:

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

We might rewrite it as

$$df = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy$$

Here, we read df as the change in the function value and dx and dy as the change in x and y. I am slightly annoyed with the textbook for making a special point of this because it really isn't anything new–it's just the equation for the tangent plane with different notation. This is really just a way of expressing it that emphasizes the change in the value of the function rather than the value of the tangent plane. We call this expression the **total differential** of f. This does give an especially convenient framework for thinking of partial derivatives as sensitivities though. As the example in the book goes, the volume of a cylinder is

$$V = \pi r^2 h$$

This means

$$dV = (2\pi rh)dr + (\pi r^2)dh$$

We can plug in specific values of r and h to determine when the volume of a cylinder is more sensitive to changes in r and when it is more sensitive to changes in h.

If we are looking at change in a specific direction, we are now talking about a real-valued function of one variable, so the linearization is a tangent line, and the total differential is

$$df = ads$$

for the appropriate number a. Hopefully you guessed that the number that goes in the place of a is

$$D_{\mathbf{u}}f$$

where **u** gives the direction that we are going. To summarize, we have

$$df = D_{\mathbf{u}}f ds$$

4 Error Analysis

We finally give some discussion to how close the linearization of a function is to the actual function. To do this, we consider the linearization $L(x, y)$ and the actual function $f(x, y)$. The difference between them is the error in the approximation:

$$E(x, y) = f(x, y) - L(x, y)$$

The book gives the result

$$|E(x, y)| \leq \frac{1}{2}M(|x - x_0|^2 + |y - y_0|^2)$$

Where M is any upper bound on the absolute values of f_{xx}, f_{yy}, f_{zz}

Example 1. *Find an upper bound for the error when approximating the function*

$$f(x, y) = x^2 - xy + \frac{1}{2}y^2 + 3$$

with its tangent plane at $(3, 2)$ *over the rectangle given by*

$$|x - 3| \leq 0.1, |y - 2| \leq 0.1$$

5 More Examples

These examples will be worked if there is extra time.

Example 2. *Find the tangent plane and normal line at the given point for the given surface.*

$$x^2 - xy - y^2 - z = 0, \quad P_0(1, -1, 3)$$

Example 3. *Find the total differential for the previous example and find which variable f is most sensitive to.*

Example 4. *Find the error from using the tangent plane approximation from the previous to examples over the rectangle*

$$|x - 1| \leq 0.1, \quad |y + 1| \leq 0.1$$

Example 5. *Find a parameteric equation for the line tangent to the curve of intersection of the surfaces at the given point.*
Surfaces: $x + y^2 + z = 2 \quad y = 1$
Point: $(\frac{1}{2}, 1, \frac{1}{2})$