We have already learned about partial derivatives and the gradient vector. A partial derivative with respect to $x$, for example, gives the rate of change in the function as we travel at unit speed in the positive $x$ direction. We have also learned about the chain rule, which we can use to talk about the derivative of a function of several variables composed with a vector-valued function. We are often interested in the rate of change in the function value as we move in an arbitrary direction (rather than just positive $x$, positive $y$, or positive $z$) at unit speed. We can use the chain rule to get this information. This is the topic of directional derivatives. We look at the following topics.

- Review of Gradient and Gradient Rules
- Moving in an arbitrary direction
- Formula for Directional Derivative
- Direction of Maximal Change
- Level Curves and Tangent Lines

1 Review of Gradient and Gradient Rules

We have already learned that the gradient is the vector given by the partial derivatives of function multiple variables. If $f$ is a function of three variables, 

$$
\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \end{pmatrix}
$$

Or if $f$ is a function of two variables,

$$
\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \end{pmatrix}
$$

The gradient follows the following differentiation rules. Let $k$ be a constant and $f$ and $g$ be functions. Then

- $\nabla (kf) = k \nabla f$
- $\nabla (f + g) = \nabla f + \nabla g$
• \( \nabla(f - g) = \nabla f - \nabla g \)
• \( \nabla(fg) = f\nabla g + g\nabla f \)
• \( \nabla \left( \frac{f}{g} \right) = \frac{g\nabla f - f\nabla g}{g^2} \)

2 Moving in an arbitrary direction

Now we come to the question of moving in an arbitrary direction. Let’s say we have \( f(x, y) = e^x \sin y \). We want to start at the point \((1, 1)\) and move to the point \((2, 3)\) at unit speed and measure the rate of change in \( f \) as we do so. We can find a unit-speed vector-valued function that goes from \((1,1)\) to \((2,3)\) and consider \( f \) composed of this function. We can take the derivative of the composition using the chain rule.

First, we find our vector-valued function. To go from \((1,1)\) to \((2,3)\), we take the function
\[
\mathbf{r}(t) = t \langle 2, 3 \rangle + (1 - t) \langle 1, 1 \rangle = \langle 1, 1 \rangle + t \langle 1, 2 \rangle
\]
This has derivative
\[
\mathbf{r}'(t) = \langle 1, 2 \rangle
\]
with speed \( \sqrt{5} \). So to make the path unit-speed, we take
\[
\mathbf{r}(t) = \langle 1, 1 \rangle + \frac{t}{\sqrt{5}} \langle 1, 2 \rangle
\]
Now we are moving in the desired direction at unit speed. To get our rate of change in the function value, we take
\[
\frac{d}{dt} f(\mathbf{r}(t)) = \nabla f \cdot \mathbf{r}'(t) = f_x x'(t) + f_y y'(t) = \frac{e^x \sin y}{\sqrt{5}} + \frac{2e^x \cos y}{\sqrt{5}}
\]
We would then substitute in \( x = y = 1 \) since that is the point we are looking at, and this would give the rate of change as we move in that direction. This is called the directional derivative.

3 Formula for Directional Derivative

In general, if \( \mathbf{u} \) is a unit vector, then the directional derivative for \( \mathbf{u} \), denoted by \( D_uf \), is the rate of change as we move at unit speed in the direction given by \( \mathbf{u} \). The formula for it is
\[
D_uf = \nabla f \cdot \mathbf{u}
\]
We can remember this by thinking of taking the derivative of
\[
f(x + t\mathbf{u})
\]
with respect to time. Note that the direction vector for traveling the positive \( x \) direction is just \((1,0,0)\), so we get the partial derivative with respect to \( x \) when we take the dot product in this case. The same holds for \( y \) and \( z \).

Since \( \|\mathbf{u}\| = 1 \), \((\mathbf{u} \text{ is a unit vector,})\) we have from the definition of dot product that
\[
D_uf = \nabla f \cdot \mathbf{u} = \|\nabla f\| \|\mathbf{u}\| \cos \theta = \|\nabla f\| \cos \theta
\]
where \( \theta \) is the angle between the gradient vector and the direction that we are interested in.
4 Direction of Maximal Change

Looking at the formula again,
\[ D_u f = \|\nabla f\| \cos \theta \]
we might ask what directions maximize or minimize the directional derivative. We know that \( \cos(0) = 1 \) and \( \cos(\pi) = -1 \), so these are the directions that maximize and minimize the directional derivative. In particular, the gradient vector gives the direction of maximum increase in the function, and the opposite direction, (the gradient times negative one,) gives the direction of greatest decrease. This also means that the magnitude of the gradient vector is the largest directional derivative at the point, and the magnitude of the gradient vector times negative 1 is the lowest possible directional derivative from that point.

**Example 1.** For the function \( f(x, y, z) = x^2 \cos(y) \sin(z) \), find the direction of maximal increase from the point \((3, \frac{\pi}{2}, \frac{3\pi}{2})\).

The gradient in this case is given by
\[ \nabla f = \langle 2x \cos(y) \sin(z), -x^2 \sin(y) \sin(z), x^2 \cos(y) \cos(z) \rangle = \langle 0, 9 \rangle \]
so the positive y direction is the direction of greatest increase and 9 is the directional derivative in that direction. The negative y direction would be the direction of greatest decrease, with a directional derivative of -9.

**Example 2.** Consider the function \( f(x, y) = x^3 y^4 \). Find the direction of greatest increase and the derivative in that direction for arbitrary \((x, y)\).

The gradient is
\[ \nabla f = \langle 3x^2 y^4, 4x^3 y^3 \rangle \]
For a specific value of \( x \) and \( y \) we would plug these values into the gradient to get the direction of maximal increase. For example, at \((1, 1)\), the gradient is
\[ \nabla f(1, 1) = (3, 4) \]
The magnitude, \( \sqrt{9 + 16} = 5 \), is the directional derivative for that direction.

5 Level Curves and Tangent Lines

If \( r(t) \) gives a level curve of the function \( f \), that is, \( f(r(t)) = c \) for some constant \( c \), then the derivative (and hence the tangent line) of the level curve must be perpendicular to the gradient of \( f \) at every point.
\[ \frac{d}{dt} f(r(t)) = \nabla f \cdot r'(t) = 0 \]
Remember, a level curve is always perpendicular to the gradient.

We can use this information to find the tangent line to a level curve through a point. If \( f(x_0, y_0) = c \), then the tangent line to the level curve for \( c \) at \((x_0, y_0)\) is given by
\[ f_x(x - x_0) + f_y(y - y_0) = 0 \]
Here, \((x, y)\) is a point on the tangent line. The vector going to this point from \((x_0, y_0)\) is perpendicular to the gradient vector, hence the equation. (We take the dot product between the two vectors and it has to be zero since they are perpendicular.)