The Inner Ear: The Basilar Membrane as a Harmonic Oscillator

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Goal: To model the cochlea in the inner ear to study the frequency response of the basilar membrane at different distances from the stapes (a small bone in the inner ear).

Further Applications: To better understand hearing loss, especially frequency-related hearing loss, and designing an artificial ear.
The Three Parts of the Ear

- **Outer ear**: Made of cartilage, this part of our ear accounts for our ability to determine if sounds come from above or below, in front or behind.

- **Middle ear**: Transmits sound vibrations from the eardrum to the cochlea. Concentrates the energy on the oval window to match the impedance to the higher impedance of the cochlea.

- **Inner ear**: The cochlea is the principal organ in the inner ear for hearing. It is a tube, about 35 mm long, separated into three compartments that are filled with fluid and twisted into a spiral.
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The Middle and Inner Ear
The Inner Ear

Reference: www.medscape.com
Frequency Tuning

The function of the cochlea is to identify the constituent frequencies of a sound wave, and thereby identify the sound. In mammals, the basilar membrane acts as the frequency analyzer.

- Vibrations of the stapes set up a wave with a particular shape on the basilar membrane.
- The vibrations create an envelope of waves whose amplitude is initially increasing, then decreasing.
- The position of the peak of the envelope is dependent on the frequency of the stimulus.

Low-frequency stimuli have a wave envelope that peaks closer to the apex of the cochlea, and high-frequency stimuli peak closer to the base.
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Frequency Response of the Basilar Membrane

Figure 23.3  Membrane waves and their envelope in the cochlea. The solid lines show the deflection of the basilar membrane at successive times, denoted (in order of increasing time) by 1, 2, 3, 4. The dashed line is the envelope of the membrane wave, and remains constant over time. (von Békésy, 1960, Fig. 12-17.)

Reference: von Békésy, 1960. Note: Base is at the stapes, where the wave originates.
Frequency Response of the Basilar Membrane

The Basilar Membrane

The basilar membrane is not a true membrane: it is not under tension.

The resistance to movement comes from the bending elasticity. The stiffness decreases exponentially from the base to the apex, with a length constant of about 7 mm.

The fluid surrounding the basilar membrane is *inviscid* and *incompressible*.

We will derive the equations of motion for the fluid in the cochlea, which are of small amplitude.
Equations of Fluid Motion

Let \( \mathbf{u} = (u_1, u_2, u_3) \) be the fluid velocity in the cochlea.
Let \( p \) = pressure.
Let \( \rho \) = density of the fluid (constant).

The mass of fluid in a fixed volume \( V \) can change only in response to fluid flux across the boundary of the volume.
Letting \( S \) be the surface of \( V \) and \( \mathbf{n} \) the outward normal,

\[
\frac{d}{dt} \int_V \rho \, dV = - \int_S \rho \, \mathbf{u} \cdot \mathbf{n} \, dS = 0
\]
Equations of Fluid Motion

The momentum of the fluid in a fixed domain \( V \) can change only in response to applied forces or to the flux of momentum across the boundary of \( V \).
Thus, for an inviscid fluid, conservation of momentum implies

\[
\frac{d}{dt} \int_V \rho u_i \, dV = - \int_S [(\mathbf{u} \cdot \mathbf{n}) \rho u_i + p n_i] \, dS
\]

By the divergence theorem,

\[
\int_V \rho \frac{\partial u_i}{\partial t} + \rho \nabla \cdot u_i u + \frac{\partial p}{\partial x_i} \, dV = 0, \quad \int_V \nabla \cdot \mathbf{u} \, dV = 0
\]
Equations of Fluid Motion

Since $V$ is arbitrary,

$$\rho \frac{\partial u}{\partial t} + \rho (\nabla \cdot u) u + \nabla p = 0$$

$$\nabla \cdot u = 0$$

If the fluid motions have small amplitude (as in the cochlea), nonlinear terms may be neglected:

$$\rho \frac{\partial u}{\partial t} + \nabla p = 0, \quad \nabla \cdot u = 0$$

If we can express $u$ as the gradient of a potential: $u = \nabla \phi$, then

$$\rho \frac{\partial \phi}{\partial t} + \tilde{p} = 0, \quad \Delta \phi = 0$$
A Harmonic Oscillator

Model the cochlea as having two rectangular compartments filled with fluid separated by the basilar membrane. (The upper compartment is the scala vestibuli and the lower compartment is the scala tympani). We now have a 2-D model in $x$ and $y$.

Each point of the basilar membrane is modeled as a simple damped harmonic oscillator with mass \( m(x) \), damping coefficient \( r(x) \), and stiffness (Hooke’s constant) \( k(x) \) that vary along the length of the membrane.

Let \( \eta(x, t) \) denote the displacement of the membrane at the distance \( x \) along the membrane. Then

\[
\begin{align*}
\frac{\partial^2 \eta}{\partial t^2} + \frac{\partial \eta}{\partial t} + k(x) \eta &= p_2(x, 0, t) - p_1(x, 0, t) \\
\end{align*}
\]

where we assumed the driving force of the pressure difference can be taken at \( y = 0 \) rather than \( y = \eta \).
Note that \( \frac{\partial \phi}{\partial y} \) is the \( y \) component of the fluid velocity. Thus the BC’s on the basilar membrane are

\[
\frac{\partial \eta}{\partial t} = \frac{\partial \phi_1}{\partial y} = \frac{\partial \phi_2}{\partial y}, \quad y = 0, \quad 0 < x < L
\]

Assuming there is no vertical motion at the top

\[
\frac{\partial \phi_1}{\partial y} = 0, \quad y = l, \quad 0 < x < L
\]

We will assume that the motion of the stapes in contact with the oval window determines the position of the oval window.
Boundary Conditions

Since $\frac{\partial \phi}{\partial x}$ is the $x$ component of the fluid velocity, the BC at $x = 0$ is

$$\frac{\partial \phi_1}{\partial x} = \frac{\partial F(y, t)}{\partial t}, \quad 0 < y < l$$

where $F(y, t)$ is the specified horizontal displacement of the oval window.

We assume there is no horizontal motion at the far end, so that at $x = L$,

$$\frac{\partial \phi_1}{\partial x} = 0, \quad 0 < y < l$$
Due to the symmetry of the problem, we seek solutions that are odd in $y$, and so we can consider only the upper region and drop the subscript 1.

When this input is at a single fixed frequency $\omega$, $F(y, t) = F(y) e^{i\omega t}$ and $\phi(x, y, t) = \hat{\phi}(x, y, \omega) e^{i\omega t}$.

If we look for solutions of this form in $\rho$ and $\eta$ as well, we get the system

\[
\Delta \hat{\phi} = 0 \quad \text{in the upper region}
\]
\[
\hat{\rho} + i\omega \rho \hat{\phi} = 0 \quad \text{in the upper region}
\]
Boundary Conditions

Defining the impedance $Z = i\omega m + r + k/(i\omega)$ and $U_0 = i\omega F(y)$, the BC's are

$$\frac{\partial \hat{\phi}}{\partial y} = i\omega \hat{\eta}, \quad i\omega \hat{\eta}Z = -\hat{p}, \quad y = 0$$

$$\frac{\partial \hat{\phi}}{\partial x} = U_0, \quad x = 0$$

$$\frac{\partial \hat{\phi}}{\partial x} = 0, \quad x = L$$

$$\frac{\partial \hat{\phi}}{\partial y} = 0, \quad y = l$$
Nondimensionalize the Equations

Scaling $x$ and $y$ by $L$, scaling $Z$ by $i\omega \rho L$, and scaling $\hat{\phi}$ by $U_0 L$, rearranging terms and dropping hats results in

$$\Delta \phi = 0 \text{ in the upper region}$$

$$\frac{\partial \phi}{\partial y} = \frac{2\phi}{Z}, \quad y = 0$$

$$\frac{\partial \phi}{\partial x} = 1, \quad x = 0$$

$$\frac{\partial \hat{\phi}}{\partial x} = 0, \quad x = 1$$

$$\frac{\partial \hat{\phi}}{\partial y} = 0, \quad y = l/L$$
Separation of variables and Fourier series result in a solution of the form

\[ \phi = x \left( 1 - \frac{x}{2} \right) - \frac{l}{L} y \left( 1 - \frac{yL}{2l} \right) + \sum_{n=0}^{\infty} A_n \cosh[n\pi (l/L - y)] \cos(n\pi x) \]

The BC on \( y = 0 \) is used to determine \( A_n \) and the plot of the function results in the correct shape and frequency-dependent behavior.
Results

Show Figure 23.7 of Keener and Sneyd.

Other approaches include using a shallow-water approximation for the equations of fluid motion or alternatively, a deep-water approximation.

Artificial cochlear implants exist. See, for example,
http://diwww.epfl.ch/lami/team/vschaik/eap/cochlea.html
http://www.virtualworldlets.net/Resources/Hosted/Resource.php?Name=TrueArtificialCochlea
Artificial Cochlea