

Homogeneous interpolation on ten points

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Abstract

In this paper we prove that for all pairs (d, m) with $d/m \geq 174/55$, the linear system of plane curves of degree d with ten general base points of multiplicity m has the expected dimension.

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Introduction

Let us denote by $\mathcal{L}(d; m_1^{s_1}, \dots, m_h^{s_h})$ the linear system of plane curves of degree d with s_i general points of multiplicity at least m_i , $i = 1, \dots, h$. This linear system has been under study for more than a century, and in general the computation of its dimension, which is the basic problem in multivariate Hermite interpolation, is still an open problem.

For nine or fewer points the dimension is known, which is a classical result going back to Castelnuovo [1] (see also [21] and [15]). There is a set of related conjectures about this dimension due to Segre, Harbourne, Gimigliano and Hirschowitz (see §1) and Nagata (see [20]), which we will review in the next section. However, even in the *homogeneous* case with ten points, i.e. for the linear system $\mathcal{L}(d; m^{10})$, the dimension is not known; it is expected to be

$$e(\mathcal{L}(d; m^{10})) = \max\left\{-1, \frac{d(d+3)}{2} - 5m(m+1)\right\}$$

and the conjectures, in this case, assert that the dimension is as expected. In this paper we prove the following result which supports the above conjectures.

Theorem 0.1. *For every pair of integers (d, m) with $d/m \geq 174/55$, the linear system $\mathcal{L}(d; m^{10})$ has the expected dimension.*

Our proof relies on a fairly complete understanding of linear systems on surfaces supporting an anticanonical curve; in §1 we review the results we need, whereas we devote §2 to the connection between the conjecture of Segre, Harbourne, Gimigliano and Hirschowitz and the one of Nagata.

We note that the problem of computing the dimension of these linear systems may be formulated on the blow-up of the plane at the multiple points; if E_i is the blow-up of the point p_i , then the system $\mathcal{L}(d; m_1^{s_1}, \dots, m_r^{s_r})$ of plane curves corresponds to $H^0(B, \mathcal{O}_B(dH - \sum_i s_i E_i))$ where B is the blow-up of the plane, and H is the pullback of the line class. In terms of the cohomology of this sheaf, the system is non-empty if $H^0 \neq 0$, and has the *expected dimension* if $H^1 = 0$: we say in this case that the system is *non-special*.

In §3 we present a degeneration of the blown-up plane which we will use to begin the analysis of the H^0 . This degeneration has been introduced in [3] and [4] for similar purposes, and we will review it briefly.

Sections 4, 5 and 8 introduce technical tools that are used in the remaining sections, in which we discuss refinements of the degeneration, necessary to prove the theorem under various hypotheses for the ratio d/m . The proof eventually consists in showing non-speciality in certain limit situations. The degenerations that we introduce can be considered to be inspired by the minimal model program in birational geometry; we perform explicit modifications (to the central fiber of our degeneration) in order to make the limit bundle more nef.

The basic technique can in principle be applied even beyond the bound 174/55. This would require however the understanding of more and more complicated degenerations, which, at the moment, seem difficult to handle.

A corollary of our theorem is that the linear system $\mathcal{L}(174; 55^{10})$ is empty, which itself has been an open problem for about twenty years (“le cas inviolé” according to A. Hirschowitz [18]). In conclusion, we would like to stress that, more than this specific result, it is the success of the technique we introduce here which provides the strongest evidence to date for the truth of the conjecture, since, as we said, there seems to be no theoretical obstruction to carrying it further, but only computational complications.

1 The Segre–Harbourne–Gimigliano–Hirschowitz conjecture

Let p_1, \dots, p_k be points of the complex projective plane \mathbb{P}^2 . We will suppose that they are distinct or *infinitely near*, i.e., we have a sequence of morphisms $f_i : S_i \rightarrow S_{i-1}$, $i = 1, \dots, k$, with $S_0 = \mathbb{P}^2$ and $f_i : S_i \rightarrow S_{i-1}$ the blow-up of S_{i-1} at a point p_i . We will set $\bar{\mathbb{P}} = S_k$. On $\bar{\mathbb{P}}$ we have (-1) -cycles E_1, \dots, E_k which are the pull-backs to $\bar{\mathbb{P}}$ of the points p_1, \dots, p_k , and the class H which is the pull-back of a line in \mathbb{P}^2 . Note that H, E_1, \dots, E_k freely generate $\text{Pic}(\bar{\mathbb{P}})$.

Let d, m_1, \dots, m_k be non-negative integers. We will denote by $\mathcal{L}(d; m_1 p_1, \dots, m_k p_k)$ the complete linear system $|dH - \sum_{i=1}^k m_i E_i|$ on $\bar{\mathbb{P}}$. We will use the same notation to denote the corresponding line bundle on $\bar{\mathbb{P}}$, as well as the push-forward of $|dH - \sum_{i=1}^k m_i E_i|$ to \mathbb{P}^2 , i.e. the linear system of plane curves of degree d with multiplicity at least m_i at p_i , $i = 1, \dots, k$. Notice that $|-K_{\bar{\mathbb{P}}}| = \mathcal{L}(3; 1^k)$.

The *virtual dimension* of $\mathcal{L}(d; m_1 p_1, \dots, m_k p_k)$ is

$$v = v(\mathcal{L}(d; m_1 p_1, \dots, m_k p_k)) = \frac{d(d+3)}{2} - \sum_{i=1}^k \frac{m_i(m_i+1)}{2}$$

and the *expected dimension* is

$$e(\mathcal{L}(d; m_1 p_1, \dots, m_k p_k)) = \min\{-1, v\}.$$

One has

$$\dim(\mathcal{L}(d; m_1 p_1, \dots, m_k p_k)) \geq e(\mathcal{L}(d; m_1 p_1, \dots, m_k p_k))$$

and $\mathcal{L}(d; m_1 p_1, \dots, m_k p_k)$ is said to be *special* if strict inequality holds. A special system $\mathcal{L}(d; m_1 p_1, \dots, m_k p_k)$ is not empty, and $h^1(\bar{\mathbb{P}}, \mathcal{L}(d; m_1 p_1, \dots, m_k p_k)) > 0$. We note that $h^2(\bar{\mathbb{P}}, \mathcal{L}(d; m_1 p_1, \dots, m_k p_k)) = 0$ if, as we will assume, $d \geq 0$.

The dimension of $\mathcal{L}(d; m_1 p_1, \dots, m_k p_k)$ is upper semicontinuous in the position of the points p_1, \dots, p_k . Therefore one may expect to have special systems for special positions of p_1, \dots, p_k , which is the case.

By contrast, if the points p_1, \dots, p_k are distinct and sufficiently general in $(\mathbb{P}^2)^k$ then the dimension of $\mathcal{L}(d; m_1 p_1, \dots, m_k p_k)$ reaches a minimum. When p_1, \dots, p_k are general we will denote $\mathcal{L}(d; m_1 p_1, \dots, m_k p_k)$ simply by $\mathcal{L}(d; m_1, \dots, m_k)$ and we will use the notation $\mathcal{L}(d; m_1^{s_1}, \dots, m_h^{s_h})$ for repeated multiplicities. A naive conjecture would be that this minimum

dimension coincides with the expected dimension. This is well known to be false; a source of counterexamples is the following. Suppose $\mathcal{L} = \mathcal{L}(d; m_1, \dots, m_k)$ is not empty. Let E be a smooth rational curve on $\bar{\mathbb{P}}$ with $E^2 = -h$, i.e. a $(-h)$ -curve. Assume that E is a (-1) -curve and that $E \cdot \mathcal{L} = -n$ with $n \geq 2$. Then nE sits in the base locus of \mathcal{L} and $h^1(\bar{\mathbb{P}}, \mathcal{L}) \geq \binom{n}{2}$. A system \mathcal{L} like this is said to be (-1) -special. Examples of this sort are $\mathcal{L}(2; 2^2)$, $\mathcal{L}(4; 2^5)$ etc.

We can state now the following conjecture due to Harbourne, Gimigliano and Hirschowitz (see [14], [13], [18]).

Conjecture 1.1 (HGH). *A system $\mathcal{L}(d; m_1, \dots, m_k)$ is special if and only if it is (-1) -special.*

There is an earlier conjecture due to B. Segre (see [24]), which, as shown in [6], is equivalent to the previous one.

Conjecture 1.2 (Segre). *If a system $\mathcal{L}(d; m_1, \dots, m_k)$ is special then its general member is non-reduced.*

We will refer to either one of the two above conjectures as to the Segre–Harbourne–Gimigliano–Hirschowitz (SHGH) conjecture.

In the homogeneous case $m_1 = \dots = m_k = m$ with $k \geq 10$, this conjecture implies (see [4]) the following.

Conjecture 1.3. *If $k \geq 10$ a system $\mathcal{L}(d; m^k)$ is never special.*

It is useful to give a different equivalent formulation of Conjecture 1.1, involving Cremona transformations. Consider a non-empty system $\mathcal{L}(d; m_1 p_1, \dots, m_k p_k)$ and assume that $d \geq m_1 \geq m_2 \geq \dots \geq m_k$. The system will be said to be *Cremona reducible* if $m_1 + m_2 + m_3 > d$ and there is an irreducible conic passing through p_1, p_2, p_3 (which is certainly the case if p_1, p_2, p_3 are distinct and not collinear). The reason for the name is that the quadratic transformation based at p_1, p_2, p_3 sends $\mathcal{L}(d; m_1 p_1, \dots, m_k p_k)$ to a new system $\mathcal{L}(d'; m'_1 p'_1, \dots, m'_k p'_k)$ with $d' = 2d - (m_1 + m_2 + m_3) < d$, and the dimension of the two systems are the same. If $m_1 + m_2 + m_3 \leq d$ the system is said to be *standard*.

Note the following result (see [18], [15]):

Proposition 1.4. *A standard system is not (-1) -special.*

Hence an equivalent formulation of Conjecture 1.1, thus of the SHGH conjecture, is:

Conjecture 1.5. *A standard system with general base points is not special.*

It goes back to Castelnuovo (see [1]), that Conjecture 1.1 holds if $k \leq 9$. More recent treatments can be found in [21], [13], [14], [15]. To be specific, one has the following more general results.

A standard system $\mathcal{L} = \mathcal{L}(d; m_1 p_1, \dots, m_k p_k)$ is said to be *excellent* [resp. *almost excellent*] if $\mathcal{L} \cdot K_{\bar{\mathbb{P}}} < 0$ [resp. if $\mathcal{L} \cdot K_{\bar{\mathbb{P}}} \leq 0$]. In other words, \mathcal{L} is excellent [resp. almost excellent] if

$$3d - \sum_{i=1}^k m_i > 0 \quad [\text{resp.} \quad 3d - \sum_{i=1}^k m_i \geq 0].$$

Moreover, one says that we are in the *anticanonical case* if there is a curve D in the linear system $|-K_{\bar{\mathbb{P}}}| = \mathcal{L}(3; p_1, \dots, p_k)$. If $k \leq 9$ one is in the anticanonical case. If there is a reduced and irreducible curve $D \in |-K_{\bar{\mathbb{P}}}|$, one says that we are in the *strong anticanonical case*. Then p_1, \dots, p_k are smooth points of a reduced, irreducible cubic curve in \mathbb{P}^2 . If p_1, \dots, p_k are general points and $k \leq 9$, then we are in the strong anticanonical case.

Recall that a line bundle \mathcal{L} is *nef* if for every irreducible curve C on $\bar{\mathbb{P}}$, one has $\mathcal{L} \cdot C \geq 0$. For the following result, see [15].

Proposition 1.6. *Suppose we are in the strong anticanonical case and consider a system $\mathcal{L} = \mathcal{L}(d; m_1 p_1, \dots, m_k p_k)$. Then:*

- (i) *if \mathcal{L} is standard, then \mathcal{L} is effective, i.e. $\dim(\mathcal{L}) \geq 0$;*
- (ii) *\mathcal{L} is almost excellent if and only if it is nef;*
- (iii) *if \mathcal{L} is excellent, then it is non-special.*

Remark 1.7. If $k \leq 9$ and p_1, \dots, p_k are general points, then a standard system \mathcal{L} is almost excellent, and actually excellent, unless $k = 9$ and \mathcal{L} is a multiple of the anticanonical system. Thus the SHGH conjecture follows by Proposition 1.6 in this case.

Remark 1.8. Assume that $k \geq 9$, p_1, \dots, p_k are general points, $m_i \geq m_{i+1}$ for each i , and $d \leq m_{k-2} + m_{k-1} + m_k$. Consider $\mathcal{L} = \mathcal{L}(d; m_1, \dots, m_k)$ and notice that $\mathcal{L} \cdot K_{\bar{\mathbb{P}}} \geq 0$. If $k = 9$ and \mathcal{L} is effective, then one concludes that $\mathcal{L} \cdot K_{\bar{\mathbb{P}}} = 0$, hence \mathcal{L} is a multiple of the anticanonical system, and has dimension zero. If $k \geq 10$ then, by considering only the first 9 points, the $k = 9$ analysis shows that \mathcal{L} is empty.

This last remark gives a first result:

Corollary 1.9. *If $d/m \leq 3$ then the linear system $\mathcal{L}(d; m^{10})$ is empty.*

The following two propositions allow us to prove the SHGH conjecture in its original form 1.1 in the case $k \leq 9$.

Proposition 1.10. *Suppose we are in the anticanonical case, and $\mathcal{L} = \mathcal{L}(d; m_1 p_1, \dots, m_k p_k)$ is an effective and nef system. If either $\mathcal{L} \cdot K_{\bar{\mathbb{P}}} \leq -1$ or p_1, \dots, p_k are general points (hence $k \leq 9$ since we are in the anticanonical case), then \mathcal{L} is non-special.*

Proof. This follows from Theorem III.1 from [16]. □

Proposition 1.11. *Consider an effective system $\mathcal{L} = \mathcal{L}(d; m_1 p_1, \dots, m_k p_k)$ with $k \leq 9$ and suppose we are in the strong anticanonical case. Then \mathcal{L} is not nef if and only if there is a smooth rational curve C on $\bar{\mathbb{P}}$ with $-2 \leq C^2 \leq -1$ and $\mathcal{L} \cdot C < 0$. If p_1, \dots, p_k are general points, then only the case $C^2 = -1$ is possible.*

Proof. If \mathcal{L} is not nef, there is some irreducible curve C such that $\mathcal{L} \cdot C < 0$ and $C^2 < 0$. On the other hand $C \cdot K_{\bar{\mathbb{P}}} \leq 0$. By the adjunction formula, C is rational and it is either a (-1) -curve or $C^2 = -2$. However this latter case cannot occur if p_1, \dots, p_k are general points (see [6], Corollary 5.4, or [8]). □

Corollary 1.12. *The SHGH conjecture holds for $k \leq 9$ general points.*

Proof. Let \mathcal{L} be an effective linear system. If \mathcal{L} is nef we are done by Proposition 1.10. If \mathcal{L} is not nef, then by Proposition 1.11 there are disjoint (-1) -curves E_1, \dots, E_h such that $\mathcal{L} \cdot E_i = -n_i < 0$, $i = 1, \dots, h$. If $n_i \geq 2$ for some i , then \mathcal{L} is (-1) -special. Otherwise the system $\mathcal{L}' = \mathcal{L} - (E_1 + \dots + E_h)$ is nef, effective and has the same dimension and virtual dimension as \mathcal{L} . We therefore finish by applying Proposition 1.10 to \mathcal{L}' . \square

Remark 1.13. Suppose $k \leq 9$ and the linear system $\mathcal{L} = \mathcal{L}(d; m_1, \dots, m_k)$ is non-trivial, effective and nef. Then it is fixed component free unless $k = 9$ and \mathcal{L} is a multiple of the anticanonical system. In fact one has $\mathcal{L} \cdot K_{\mathbb{P}^2} \leq 0$ and $\mathcal{L}^2 \geq 0$. If $k \leq 8$, then $\mathcal{L} \cdot K_{\mathbb{P}^2} \leq -1$ by the index theorem, and the assertion follows by Theorem III.1 from [16] (note that in case (b) of that reference, there cannot be a fixed component because of the generality of the base points). If $k = 9$, then the same argument works if $\mathcal{L} \cdot K_{\mathbb{P}^2} \leq -1$. If $\mathcal{L} \cdot K_{\mathbb{P}^2} = 0$ then \mathcal{L} is a multiple of the anticanonical system.

We close this section with a useful lemma.

Lemma 1.14. *Consider an effective system $\mathcal{L} = \mathcal{L}(d; m_1, \dots, m_k)$ with $k \leq 9$. If E is a (-1) -curve such that $E \cdot \mathcal{L} = -n < 0$, then E appears in the base locus of \mathcal{L} exactly with multiplicity n .*

Proof. Let E_1, \dots, E_h be all (-1) -curves such that $E_i \cdot \mathcal{L} = -n_i < 0$. We may assume $E = E_1$. Then E_1, \dots, E_h is a (-1) -configuration in the sense of [3], i.e. $E_i \cdot E_j = 0$ if $1 \leq i < j \leq h$. Moreover $\mathcal{L} = \sum_{i=1}^h n_i E_i + \mathcal{L}'$, where $\mathcal{L}' = \mathcal{L}(d'; m'_1, \dots, m'_k)$ is effective and not (-1) -special; hence it is non-special, and $\mathcal{L}' \cdot E_i = 0$, $i = 1, \dots, h$.

Then \mathcal{L}' is nef and by Remark 1.13 it has no (-1) -curve in its base locus. \square

2 The Nagata conjecture

The SHGH conjecture implies another famous open conjecture by Nagata (see [20] and [6]):

Conjecture 2.1 (Nagata). *The system $\mathcal{L}(d; m_1, \dots, m_k)$ is empty as soon as $k \geq 10$ and*

$$\sum_{i=1}^k m_i \geq d\sqrt{k}.$$

Nagata's conjecture holds if k is a perfect square (see [20]). In this case also SHGH conjecture holds (see [11], [7], [23]).

For homogeneous linear systems, Nagata's conjecture reads as follows.

Conjecture 2.2 (Homogeneous Nagata). *Assume $k \geq 10$. The system $\mathcal{L} = \mathcal{L}(d; m^k)$ is empty as soon as $\mathcal{L}^2 \leq 0$, i.e. as soon as*

$$d/m \leq \sqrt{k}.$$

For homogeneous linear systems with non-positive self-intersection Nagata's conjecture and the SHGH conjecture are equivalent: in this case they both predict that the system is empty. Nagata's conjecture does not directly make any prediction for homogeneous linear systems with positive self-intersection as SHGH does. However, as we are going to show next, it can be seen as an asymptotic version of the SHGH conjecture.

Proposition 2.3. *Fix a number $x \geq \sqrt{k}$ and suppose that if $\mathcal{L}(\delta; \mu^k)$ is not empty, then $\delta/\mu > k/x$. Then for all pairs (d, m) of positive numbers such that $d/m \geq x$ the linear system $\mathcal{L} = \mathcal{L}(d; m^k)$ is ample. Moreover there is an integer $N(d/m)$ such that for all $n > N(d/m)$ the linear system $\mathcal{L}(nd; (nm)^k)$ is non special.*

Proof. One has $\mathcal{L}^2 > 0$. This is clear if k is not a perfect square; if k is a perfect square, it follows from Nagata's conjecture, which holds in this case.

Suppose C is an irreducible curve on $\bar{\mathbb{P}}$ such that $\mathcal{L} \cdot C \leq 0$. Since $\mathcal{L}^2 > 0$, there is a suitable multiple of \mathcal{L} which is effective. This implies that $C^2 \leq 0$. Now uniformize C in the sense of Nagata (see [20], p. 285), thus getting the homogeneous linear system $\mathcal{L}' = \mathcal{L}(\delta; \mu^k)$ formed by the sum of C with its transforms via all permutations of the base points p_1, \dots, p_k which change C into a different curve. One has $\delta/\mu > k/x$. By monodromy, one also has $d\delta - km\mu = \mathcal{L} \cdot \mathcal{L}' \leq 0$, which leads to a contradiction.

The ampleness assertion follows by the Nakano–Moishezon Theorem.

As a consequence, there is a positive number ϵ such that all linear systems of the form $\mathcal{L}(\delta; \mu^k)$ with $\delta/\mu \geq x - \epsilon$ are big and nef.

Now consider all homogeneous linear systems of the form $n\mathcal{L} - K_{\bar{\mathbb{P}}} = \mathcal{L}(nd+3; (nm+1)^k)$. If n is large enough, we have $(nd+3)/(nm+1) \geq x - \epsilon$. Thus $n\mathcal{L} - K_{\bar{\mathbb{P}}}$ is big and nef and therefore $h^1(\bar{\mathbb{P}}, n\mathcal{L}) = 0$ by the Mumford vanishing theorem [19]. \square

Corollary 2.4. *If $k \geq 10$ and the homogeneous Nagata conjecture holds for k general points, then the ray generated by $\mathcal{L}(d; m^k)$ belongs to the effective cone if and only if it belongs to the ample cone, in which case there is an integer $N(d/m)$ such that for all $n > N(d/m)$ the linear system $\mathcal{L}(nd; (nm)^k)$ is non-special.*

On the other hand, one has the following:

Proposition 2.5. *Assume $k \geq 10$ and fix a number $x \geq \sqrt{k}$. Suppose that for all pairs (d, m) of positive numbers such that $d/m \geq x$, there is an integer n such that the linear system $\mathcal{L}(nd; (nm)^k)$ is not empty and non-special. Then all linear systems $\mathcal{L}(d; m^k)$ with $d/m \geq x$ are big and nef.*

Proof. Set $\mathcal{L} = \mathcal{L}(d; m^k)$. Replace \mathcal{L} with a multiple so as to get an effective, non-special linear system. Assume C is an irreducible curve such that $\mathcal{L} \cdot C < 0$. Then $\mathcal{L}(-C)$ is effective and the exact sequence

$$0 \rightarrow \mathcal{L}(-C) \rightarrow \mathcal{L} \rightarrow \mathcal{L}|_C \rightarrow 0$$

implies that $0 = h^1(\bar{\mathbb{P}}, \mathcal{L}) = h^1(C, \mathcal{L}|_C)$. This yields $\deg(\mathcal{L}|_C) \geq p_a(C) - 1$, which forces C to be a (-1) -curve (see [8]). But since there is no homogeneous (-1) -configuration when one blows up $k \geq 10$ points in the plane (see [4]), we find a contradiction. \square

Remark 2.6. The above proof shows that the only way a system $\mathcal{L} = \mathcal{L}(d; m^k)$ can be not ample, is because of the existence of a curve C with $p_a(C) = 1$ and $\mathcal{L} \cdot C = 0$.

As a consequence we have:

Corollary 2.7. *In the same hypotheses as in Proposition 2.5, if $\mathcal{L}(\delta; \mu^k)$ is not empty, then $\delta/\mu \geq k/x$.*

In particular, if for all pairs (d, m) of positive numbers such that $d/m \geq \sqrt{k}$, there is an integer n such that the linear system $\mathcal{L}(nd; (nm)^k)$ is not empty and non-special, then the homogeneous Nagata conjecture holds.

Proof. Set $\mathcal{L}' = \mathcal{L}(\delta; \mu^k)$. Assume $\delta/\mu < k/x$. Then we can choose positive integers d, m such that $x < d/m < k\mu/\delta$. Then the linear system $\mathcal{L}(d; m^k)$ is big and nef, but $\mathcal{L} \cdot \mathcal{L}' < 0$, a contradiction. \square

3 The first degeneration

From now on, we will consider homogeneous linear systems $\mathcal{L}(d; m^{10})$ with ten general base points. In order to show that the linear system $\mathcal{L}(d; m^{10})$ has the expected dimension, we will make an appropriate degeneration, both of the plane (blown up at the ten general points) and of the line bundle. The full analysis will require several different degenerations, depending on d, m , and their ratio d/m ; the first degeneration we will present has been described in [3] and [4].

We first consider the trival family $\Delta \times \mathbb{P}^2 \rightarrow \Delta$ over a disc Δ and blow up a point in the central fiber. We thus get a flat and proper family $\mathcal{Y} \rightarrow \Delta$ over Δ , where the general fibre Y_t for $t \neq 0$ is a \mathbb{P}^2 , and the central fibre Y_0 is reducible surface $\mathbb{P} \cup \mathbb{F}$, where $\mathbb{P} \cong \mathbb{P}^2$ is a projective plane, $\mathbb{F} \cong \mathbb{F}_1$ is a plane blown up at a point, and \mathbb{P} and \mathbb{F} meet transversally along a smooth rational curve E which is the exceptional divisor on \mathbb{F} and a line on \mathbb{P} (see Figure 1).

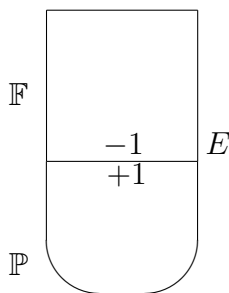


Figure 1: the degeneration of the plane

We now choose four general points on \mathbb{P} and six general points on \mathbb{F} . Consider these ten points as limits of ten general points in the general fibre Y_t and simultaneously blow these points up in the family \mathcal{Y} . This creates ten surfaces R_i , ruled over Δ , whose intersection with each fiber is a (-1) -curve, the exceptional curve for the blow-up of that point in the family. We denote by $\mathcal{X}_1 \rightarrow \Delta$ this new family. The general fibre $X_{1,t}$ for $t \neq 0$ is a plane blown up at ten general points. The central fibre $X_{1,0}$, shown in Figure 2, is the union $V_1 \cup Z_1$ where:

- V_1 is a plane blown up at four general points;

- Z_1 is a plane blown up at seven general points;
- V_1 and Z_1 meet transversally along a smooth rational curve E which is a (-1) -curve on Z_1 , whereas $E^2 = 1$ on V_1 : it is the pull-back of a line.

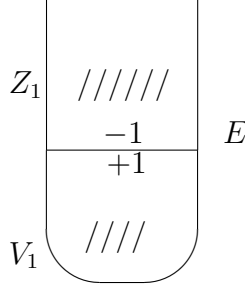


Figure 2: the degeneration of the blown-up plane

Consider the line bundle $\mathcal{L}_0 = \pi^*(\mathcal{O}_{\mathbb{P}^2}(d)) \otimes \mathcal{O}_{\mathcal{X}_1}(-\sum_i mR_i)$, where $\pi : \mathcal{X}_1 \rightarrow \mathbb{P}^2$ is the natural map. This restricts to $\mathcal{L}(d; m^{10})$ on the general fibre, whereas on the central fibre it is $\mathcal{L}(0; m^4)$ on V_1 and $\mathcal{L}(d; 0, m^6)$ on Z_1 . In this notation and in the following, the first multiplicity for bundles on Z_1 refers to the point corresponding to E .

We will further twist this bundle by a suitable multiple of Z_1 . Namely, we choose a parameter a (to be determined later), and define

$$\mathcal{L}_1 := \mathcal{L}_0 \otimes \mathcal{O}_{\mathcal{X}}((2m + a)Z_1).$$

We will denote by \mathcal{L}_{V_1} and \mathcal{L}_{Z_1} the restrictions of \mathcal{L}_1 to V_1 and Z_1 ; these bundles have the form

$$\mathcal{L}_{V_1} = \mathcal{L}(2m + a; m^4), \quad \mathcal{L}_{Z_1} = \mathcal{L}(d; 2m + a, m^6).$$

Using this degeneration we have (compare with [16], Lemma II.7):

Proposition 3.1. *If $d/m \geq 10/3$ then $\mathcal{L}(d; m^{10})$ has the expected dimension.*

Proof. We set $a = 0$ for this analysis. We will use semi-continuity and the transversality of the restricted linear systems to the double curve E (see [3], Proposition 3.3, (b)). Note that these restricted systems both have degree $2m$.

First we notice that \mathcal{L}_{V_1} and \mathcal{L}_{Z_1} are both not empty and non-special.

As for $\mathcal{L}_{V_1} = \mathcal{L}(2m; m^4)$, this is the dimension m linear system whose elements are m conics through the four points. The dimension r_{V_1} of the restriction of this system to E is also m .

As for $\mathcal{L}_{Z_1} = \mathcal{L}(d; 2m, m^6)$, we apply Proposition 1.11 to prove that it is nef. The linear system $|-K_{Z_1}|$, corresponding to $\mathcal{L}(3; 1^7)$, is base point free of dimension 2 and all (-1) -curves are contained in a curve of $|-K_{Z_1}|$. The crucial computation to make is to intersect the bundle with the (-1) -curve which is the proper transform of the unique cubic curve C in $\mathcal{L}(3; 2, 1^6)$; this intersection number is $3d - 10m \geq 0$. By Proposition 1.10 the system is non-special, and it is non-empty, since the virtual dimension is positive (which is also implied by the inequality $d/m \geq 10/3$).

Let r_{Z_1} be the dimension of the restriction of this system to E . One has

$$r_{Z_1} = \dim(\mathcal{L}(d; 2m, m^6)) - \dim(\mathcal{L}(d; 2m + 1, m^6)) - 1$$

since $\mathcal{L}(d; 2m + 1, m^6)$ is the *kernel* subsystem of elements of $\mathcal{L}(d; 2m, m^6)$ that restrict to zero on E . If $d/m > 10/3$, then, similar considerations as above imply that $\mathcal{L}(d; 2m + 1, m^6)$ is not empty and non-special. Therefore r_{Z_1} can be computed as $2m$, i.e. the restricted system is complete. If $d/m = 10/3$, the cubic curve C splits off twice from the subsystem $\mathcal{L}(d; 2m + 1, m^6)$, but arguments as above show that the residual system is non-special. Hence $\mathcal{L}(d; 2m + 1, m^6)$ has speciality exactly one, and $r_{Z_1} = 2m - 1$.

Since in either case $r_{V_1} + r_{Z_1} \geq 2m - 1$, we can apply Proposition 3.3, (b) from [3] to conclude. \square

4 Throwing (-1)-curves

The reason why a relatively simple degeneration such as the one presented above will not suffice to prove the general statement is that the line bundles on the individual surfaces may become special. By the SHGH Conjecture, this should be a consequence of having (-1)-curves on those surfaces intersecting the bundle negatively.

Our technique to handle this situation will be to blow up the offending (-1)-curve, and twist by an appropriate multiple of the exceptional ruled surface. We hope to arrive at the situation where the ruled surface is a $\mathbb{P}^1 \times \mathbb{P}^1$, and can be blown down via the other ruling, contracting the original (-1)-curve. This process will create exceptional curves on the surfaces that the (-1)-curve meets. We refer to this technique in general as *throwing* the (-1)-curve. We explain this idea in two cases which will be relevant for our purposes.

4.1 A 1-Throw.

To be specific, suppose that E is a (-1)-curve on one of the components V of a local normal crossings semistable degeneration and that the restriction \mathcal{L}_V to V of the line bundle \mathcal{L} on the threefold has the property that $\mathcal{L}_V \cdot E = -k < 0$. Suppose further that there is a double curve R where V meets another component Z , and E meets transversally that curve R at a single point p and meets no other double curve.

Blow up the curve E , obtaining the ruled surface S ; by the Triple Point Formula (cf. [22], [12]) S will be isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, and will meet V along E , which is one of the members of one of the rulings on S . This blow-up will effect a blow-up of the other surface Z at the point p , creating a new (-1)-curve E' on the blow-up Z' ; the surface S also meets Z' along E' , which is a member of the other ruling on S (see the left hand side of Figure 3).

Note that the normal bundle of S in the threefold has bidegree $(-1, -1)$, and that the pullback of the bundle \mathcal{L} , restricted to S , has bidegree $(-k, 0)$.

Replace the line bundle \mathcal{L} by $\mathcal{L}' = \mathcal{L} \otimes \mathcal{O}(-kS)$. The restrictions of \mathcal{L}' to the various components are as follows: $\mathcal{L}'|_V = \mathcal{L}_V(-kE)$; $\mathcal{L}'|_{Z'} = \pi^*(\mathcal{L}_Z)(-kE')$; $\mathcal{L}'|_S$ has bidegree $(0, k)$. Thus the new bundle on the surface V does not meet E anymore.

With this we see that we may blow S down to E' via the other ruling, obtaining an alternate degeneration; this will blow down the original (-1)-curve E as desired, and retain

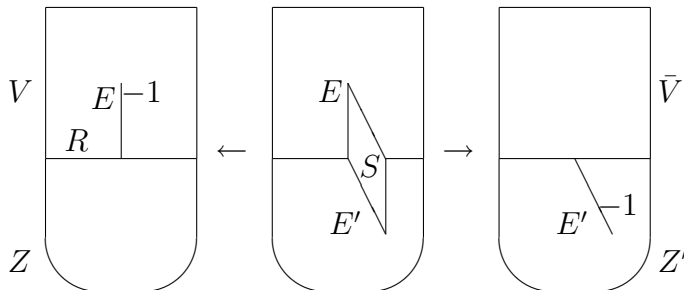


Figure 3: throwing a (-1) -curve

the blow-up Z' of the surface Z . The surface V is blown down to \bar{V} , and the bundle on \bar{V} is simply the bundle on V , with E removed. On Z' the bundle is the pullback of the original bundle on Z , twisted by $-k$ times the exceptional divisor E' . This corresponds to adding a point of multiplicity k to the system on Z (see the right hand side of Figure 3).

This operation will be referred to as a *1-throw* (of E on V).

4.2 A 2-Throw.

Let us now consider the case when the (-1) -curve E meets transversally the double curve locus R in two points p_1 and p_2 . We still assume that E lies on the component V and that the restricted system \mathcal{L}_V has the property that $\mathcal{L}_V \cdot E = -k < 0$. Again blow up E , obtaining the ruled surface T , which is isomorphic to \mathbb{F}_1 by the Triple Point Formula; T meets V along E , and this is also the (-1) -curve which is the negative section of T . The blow-up will create a blow-up Z' of the surface (or surfaces) Z that meet V along R , at the two points p_1 and p_2 , with two exceptional divisors G_1 and G_2 on Z' . These G_i are fibers of the ruling of T . This is shown on the left side of Figure 4.

Now blow up E again, creating the ruled surface S . This time $S \cong \mathbb{P}^1 \times \mathbb{P}^1$; S meets V along E , and it meets T along the negative section. The blow-up effects a further two blow-ups on Z' , creating the surface Z'' , and two more exceptional divisors F_1 and F_2 respectively, which are (-1) -curves on Z'' . By abusing notation we denote by G_1, G_2 their proper transforms on Z'' ; these are now (-2) -curves. The surface S now occurs with multiplicity two in the central fiber of the degeneration, since it was obtained by blowing up a double curve, and its normal bundle in the total space of the degeneration has bidegree $(-1, -1)$. All this is shown in the central part of Figure 4.

Write $k = 2\ell - \epsilon$, with $\epsilon \in \{0, 1\}$, and replace the line bundle \mathcal{L} by $\mathcal{L}' = \mathcal{L} \otimes \mathcal{O}(-kS) \otimes \mathcal{O}(-\ell T)$. The restrictions of \mathcal{L}' to the various components are as follows: $\mathcal{L}'|_V = \mathcal{L}_V(-kE)$; $\mathcal{L}'|_{Z''} = \pi^*(\mathcal{L}_Z)(-k(F_1 + F_2) - (\ell)(G_1 + G_2))$; $\mathcal{L}'|_T = \epsilon H$, where H is the line class on $T \cong \mathbb{F}_1$; $\mathcal{L}'|_S$ has bidegree $(0, \ell - \epsilon)$.

As in the case of the 1-throw, we may now blow S down the other way. This contracts E on the surface V , thus creating a new surface \bar{V} , and contracts the negative section of T , so that T becomes a \mathbb{P}^2 . The image Z''' of the surface Z'' has the two curves F_1 and F_2 identified (see the rightmost side of Figure 4, where, by abusing notation, we still denote by T its image after the contraction of S).

The bundle on the new plane created by T has degree ϵ . The bundle on Z''' can be

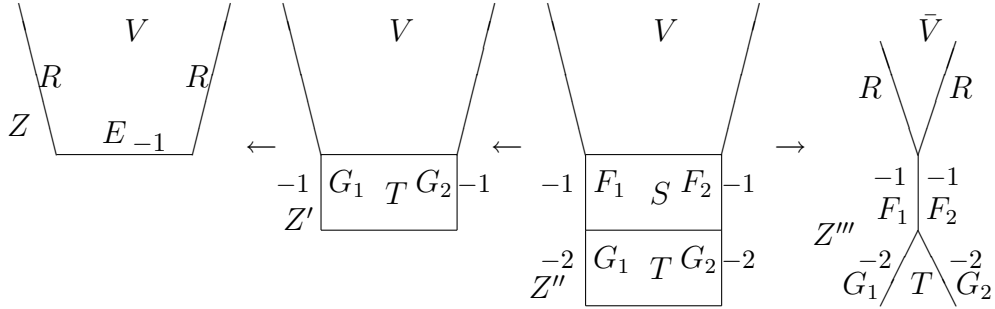


Figure 4: throwing a (-2) -curve

interpreted in the geometry of Z where two new *compound multiple points* have been created, each one a point of multiplicity ℓ and an infinitely near point of multiplicity $\ell - \epsilon$. We will denote this type of compound multiple point by the notation $[m_1, m_2]$, thus indicating a multiple point m_1 and an infinitely near multiple point m_2 . Thus the above process produces two $[\ell, \ell - \epsilon]$ points on Z .

We refer to this operation as a *2-throw* (of E on V).

It is worth pointing out that the contraction of S which results in the identification of F_1 and F_2 on Z'' will force us to take this into account when we will make the analysis of the linear systems on the degenerations.

Although one may imagine more complicated throws (when the (-1) -curve E meets the double curve in more than two points) we will not require such constructions in the sequel of this paper.

Note that in a 2-throw, if the two points p_1 and p_2 lie on the same component of the double curve R , then the surface Z is a single component, the curve R becomes, after the 2-throw, a nodal curve, and the construction results in a non-normal component of the degeneration, because of the identification of F_1 and F_2 . However this presents no real problems in the analysis; the central fiber still has local normal crossings, and all linear system computations on the various components can be done on their normalizations.

5 Computation of the limit dimension

We will next perform a series of throws of (-1) -curves starting from the first degeneration described in section §3. This will create more complicated degenerations of the blown up plane, which will have more than two components, but still with local normal crossings and semistable. These degenerations will carry a suitable limit of the relevant line bundle, and it is our task to compute the dimension of the space of sections of the limit bundle to show that it is equal to the expected dimension. Then, by appealing to semicontinuity, we will prove non-speciality of the bundle on the general surface.

Since we will have more than two components in the degeneration, we cannot appeal directly to Proposition 3.3, (b) from [3], as we did in §3, to make the computation of the dimension in the limit. We therefore have to develop a more general analysis.

In any event, as in the case of two components, the space of limit sections is a fibre product, namely, one must give sections on the components, which agree on the double

curves.

In order to compute such a fibre product it will be convenient for us to proceed iteratively, by building up the degeneration one surface at a time. This leads to an analogue of Proposition 3.3, (b) from [3], where the involved surfaces may be reducible.

To be specific, suppose we have a (local normal crossings) surface $X_0 = V \cup W$, and a line bundle \mathcal{L} on X_0 , restricting to \mathcal{L}_V on V and to \mathcal{L}_W on W . We denote by C the intersection curve of V and W , with \mathcal{L} restricting to \mathcal{L}_C on C . Then, whether or not V and W are irreducible, $H^0(X, \mathcal{L})$ is the kernel of the difference map

$$H^0(V, \mathcal{L}_V) \oplus H^0(W, \mathcal{L}_W) \rightarrow H^0(C, \mathcal{L}_C). \quad (5.1)$$

Geometrically this reads as follows: the curves in the linear system \mathcal{L} are Cartier divisors on X_0 and, as such, they have to be the union of a curve in \mathcal{L}_V and a curve in \mathcal{L}_W , which meet C at the same points.

If we know the dimension of the three spaces involved in the map (5.1) and we also know that the difference map is surjective, then we can compute the dimension of $H^0(X, \mathcal{L})$. The hypothesis of part (b) of Proposition 3.3, from [3] is equivalent to the surjectivity in that case.

By considering the exact sequence (at the sheaf level)

$$0 \rightarrow \mathcal{L} \rightarrow \mathcal{L}_V \oplus \mathcal{L}_W \rightarrow \mathcal{L}_C \rightarrow 0,$$

we see that if $H^1(V, \mathcal{L}_V) = H^1(W, \mathcal{L}_W) = 0$, then $H^1(X_0, \mathcal{L}) = 0$ if and only if the difference map at the H^0 level is surjective.

In our case we will have $\pi : \mathcal{X} \rightarrow \Delta$ a flat, proper, semistable, local normal crossings degeneration of smooth projective surfaces X_t , $t \neq 0$, to the central fibre X_0 ; the total space \mathcal{X} is endowed with a line bundle $\mathcal{L}_{\mathcal{X}}$, restricting to \mathcal{L} on X_0 . This central fiber is a divisor in the threefold X of the form $\sum_{i=1}^n V_i$. We denote by \mathcal{L}_i the restriction of \mathcal{L} to V_i .

Set $W_k = \sum_{i=1}^k V_i$, and $\mathcal{L}^{(k)}$ the restriction of \mathcal{L} to W_k ; note that $W_k = W_{k-1} + V_k$. Denote by C_{k-1} the intersection of W_{k-1} and V_k . The considerations above apply and we can use them to compute $H^0(W_k, \mathcal{L}^{(k)})$. For $k = n$ we have the desired space of sections.

These arguments lead to the following statement:

Proposition 5.2. *In the above setting, if:*

- (i) $H^1(V_i, \mathcal{L}_i) = 0$, for all i ;
- (ii) *the difference maps $H^0(W_{k-1}, \mathcal{L}^{(k-1)}) \oplus H^0(V_k, \mathcal{L}_k) \rightarrow H^0(C_{k-1}, \mathcal{L}_{C_{k-1}})$ are surjective for all k ;*

then $H^1(W_k, \mathcal{L}^{(k)}) = 0$, for all k . Hence $H^1(X_0, \mathcal{L}) = 0$, and non-speciality of the bundle on the general surface X_t follows by semicontinuity.

Remark 5.3. The surjectivity of the difference map in (ii) will follow, in our applications, exactly as in part (b) of Proposition 3.3 from [3], from a dimension count and an appropriate transversality property, which will have to be checked case by case. However a sufficient condition for the surjectivity is that either one of the natural restriction maps is surjective, which would follow from the vanishing of the appropriate H^1 (i.e., either $H^1(W_{k-1}, \mathcal{L}^{(k-1)}(-C_{k-1})) = 0$ or $H^1(V_k, \mathcal{L}_k(-C_{k-1})) = 0$).

Remark 5.4. We will have situations in which, due to the application of a 2–throw, a single component V of X_0 is non–normal with a double curve C . However, the normalization \tilde{V} of V will be smooth and V will be obtained by identifying two non–intersecting curves C_1 and C_2 , both isomorphic to C . Denote by $\mathcal{L}_{\tilde{V}}$, \mathcal{L}_{C_i} the pull–backs of the bundle to \tilde{V} and the C_i , respectively. Then \mathcal{L}_C injects into $\mathcal{L}_{C_1} \oplus \mathcal{L}_{C_2}$, with quotient sheaf \mathcal{N} ; and \mathcal{L}_V is the kernel of the natural map $\mathcal{L}_{\tilde{V}} \rightarrow \mathcal{N}$. Therefore, if, as above, the corresponding map on global sections is surjective, and $H^1(\tilde{V}, \mathcal{L}_{\tilde{V}}) = 0$, then $H^1(V, \mathcal{L}_V) = 0$. In particular $H^0(\tilde{V}, \mathcal{L}_{\tilde{V}}) \rightarrow H^0(C, \mathcal{N})$ is surjective if

$$H^0(\tilde{V}, \mathcal{L}_{\tilde{V}}) \rightarrow H^0(C_1, \mathcal{L}_{C_1}) \oplus H^0(C_2, \mathcal{L}_{C_2})$$

and

$$H^0(C_1, \mathcal{L}_{C_1}) \oplus H^0(C_2, \mathcal{L}_{C_2}) \rightarrow H^0(C, \mathcal{N})$$

are both surjective. In our applications the latter map will be surjective because $H^1(C, \mathcal{L}) = 0$.

Alternatively, the above criterion can be deduced from the cohomology of the exact sequence

$$0 \rightarrow \mathcal{L}_V \rightarrow f_*\mathcal{L}_{\tilde{V}} \rightarrow \mathcal{N} \rightarrow 0,$$

and \mathcal{N} , as above, is supported on C .

Again, from a geometric viewpoint, a curve in \mathcal{L}_V corresponds to a curve in $\mathcal{L}_{\tilde{V}}$ which meets C_1 and C_2 in corresponding points, which are glued on V .

Note however that this is only a necessary condition: in general, curves in $\mathcal{L}_{\tilde{V}}$ meeting C_1 and C_2 in corresponding points might not correspond to curves in \mathcal{L}_V . Actually, there could be more line bundles on \tilde{V} corresponding to the same line bundle \mathcal{L}_V on V . An easy example is the following: consider the curve of arithmetic genus 1 obtained by gluing two distinct points p_1, p_2 on \mathbb{P}^1 . Then $V = \mathbb{P}^1 \times C$ is obtained from $\tilde{V} = \mathbb{P}^1 \times \mathbb{P}^1$ by gluing the two distinct fibres C_1, C_2 over p_1, p_2 . Take a non–trivial line bundle of degree 0 on C and pull it back on V , thus getting a line bundle \mathcal{L}_V . Any such bundle corresponds to the trivial bundle on \tilde{V} , no non–zero section of which descends to a section of \mathcal{L}_V .

In any event, $\dim \mathcal{L}_V$ is bounded above by the dimension of the family of curves in $\mathcal{L}_{\tilde{V}}$ meeting C_1 and C_2 in corresponding points.

6 The second degeneration: throwing the cubic

In the proof of Proposition 3.1, the hypothesis $d/m \geq 10/3$ was used in a critical way to show that the system on the surface Z_1 is nef, and in particular to show that the intersection with the cubic curve C in the system $\mathcal{L}(3; 2, 1^6)$ is non–negative. As soon as $d/m < 10/3$, this intersection becomes negative, and we propose to employ a 2–throw to remove it from Z_1 , creating a second degeneration.

We assume for this second degeneration that $16/5 \leq d/m < 10/3$. Let us write

$$d = 2c + e, \quad \text{with } e \in \{0, 1\}. \tag{6.1}$$

We return to the first degeneration, and note that if $a \geq 0$, then

$$C \cdot \mathcal{L}_{Z_1} = 3d - 10m - 2a < 0.$$

Hence C splits exactly $10m - 3d + 2a$ times from \mathcal{L}_{Z_1} . Furthermore it meets the double curve E twice, at points p_1 and p_2 .

We perform a 2-throw of C on Z_1 , blowing up C twice and contracting the second ruled surface, which is a $\mathbb{P}^1 \times \mathbb{P}^1$, the other way. Set

$$b = 5m + a - 3c - e \tag{6.2}$$

and note that $10m - 3d + 2a = 2b - e$, so that the 2-throw creates two $[b, b - e]$ -points on V_1 . This results in our second degeneration, which now consists of three surfaces, as shown in Figure 5:

- V_2 , the transform of V_1 . The normalization \tilde{V}_2 is V_1 blown up at p_1 and p_2 (twice each), creating exceptional curves F_1, F_2, G_1 , and G_2 . We have $F_i^2 = -1, G_i^2 = -2, F_i \cdot G_j = \delta_{ij}$; in addition, the F_i 's meet E transversally (at the p_i 's). The transform of the double curve E now has self-intersection -3 on the normalization. The surface V_2 is obtained from the normalization by suitably identifying F_1 and F_2 . The double curve E becomes a nodal curve. The linear system on the normalization \tilde{V}_2 has the form $\mathcal{L}_{V_2} = \mathcal{L}(2m + a; m^4, [b, b - e]^2)$.
- Z_2 , the transform of Z_1 . The surface Z_2 is smooth: it is obtained from Z_1 by blowing down the curve C . The linear system on Z_2 is obtained from the system $\mathcal{L}(d; 2m + a, m^6)$ on Z_1 by removing $10m - 3d + 2a$ times the cubic (i.e. the system $\mathcal{L}(3; 2, 1^6)$), and so has the form $\mathcal{L}(10d - 30m - 6a; 6d - 18m - 3a, (3d - 9m - 2a)^6)$.
- T_2 , the surface created by the 2-throw, which is isomorphic to a projective plane, meeting the surface V_2 along G_1 and G_2 , which are lines in T_2 . The linear system on T has degree e .

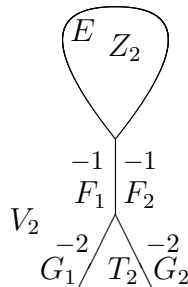


Figure 5: the second degeneration

Now let us analyze the linear systems on these three components. First we note that the linear system on T_2 is non-special.

Let us turn our attention to Z_2 , and set

$$\alpha = d - 3m. \tag{6.3}$$

Hence we may write the system on Z_2 as $\mathcal{L}(10\alpha - 6a; 6\alpha - 3a, (3\alpha - 2a)^6)$. We have the series of quadratic transformations shown in the following table: in each row the first number

indicates the degree of the system, the following numbers denote the multiplicities, and we underline the base points used in each quadratic transformation:

$$\begin{array}{cccccccc}
10\alpha - 6a; & \underline{6\alpha - 3a}, & \underline{3\alpha - 2a}, & \underline{3\alpha - 2a}, & 3\alpha - 2a, & 3\alpha - 2a, & 3\alpha - 2a, & 3\alpha - 2a \\
8\alpha - 5a; & \underline{4\alpha - 2a}, & \alpha - a, & \alpha - a, & \underline{3\alpha - 2a}, & \underline{3\alpha - 2a}, & 3\alpha - 2a, & 3\alpha - 2a \\
6\alpha - 4a; & \underline{2\alpha - a}, & \alpha - a, & \alpha - a, & \alpha - a, & \alpha - a, & \underline{3\alpha - 2a}, & \underline{3\alpha - 2a} \\
4\alpha - 3a; & 0 & \alpha - a, & \alpha - a, & \alpha - a, & \alpha - a, & \alpha - a, & \alpha - a.
\end{array} \tag{6.4}$$

The 0 as the first multiplicity represents the cubic curve, now blown down. Hence the system on Z_2 is Cremona equivalent to $\mathcal{L}(4\alpha - 3a; (\alpha - a)^6)$, which if $0 \leq a \leq \alpha$ is excellent and therefore non-special by Proposition 1.6.

We next consider the system on V_2 , or rather on its normalization \tilde{V}_2 .

Lemma 6.5. *Assume $16/5 \leq d/m < 10/3$. If $5m - 3c - e \leq a \leq d - 3m$, i.e., if*

$$b/2 \leq a \leq \alpha,$$

then the system \mathcal{L}_{V_2} is non-empty and non-special.

Proof. Recall that the linear system \mathcal{L}_{V_2} has the form $\mathcal{L}(2m + a; m^4, [b, b - e]^2)$ on the normalization \tilde{V}_2 .

We first claim that, with a in the given range, \mathcal{L}_{V_2} is effective because its virtual dimension v is non-negative. After substituting $b = 5m + a - 3c - e$, v becomes a function of a , m , c , and e ; for fixed m , c , and e , one has $\partial v / \partial a = 6\alpha - 3a - 1/2 > 0$ in the given range. Hence to see that $v \geq 0$, it suffices to check this for the left endpoint $a = 5m - 3c - e$. In this case

$$2v = (5m - 3c)(45c - 71m) + (15c - 23m) + 6e(31m - 19c - 3).$$

If $e = 0$, then $d = 2c$, and the inequalities on d/m imply that $8m \leq 5c$ and $3c < 5m$. Hence $5m - 3c \geq 1$, $45c - 71m \geq m$, and $15c - 23m \geq m$, so that $v \geq m$. If $e = 1$, then $d = 2c + 1$, and the inequalities are $16m \leq 10c + 5$ and $6c + 3 < 10m$; the first cannot be an equality for parity reasons, so that in fact $8m \leq 5c + 2$, and the second gives $3c + 2 \leq 5m$. Hence $5m - 3c \geq 2$, and $45c - 71m \geq m - 18$, so the quadratic part is at least $2m - 36$. The linear term is now $163m - 99c - 18$, and since $5m - 3c \geq 2$, we have $165m - 99c \geq 66$; hence the linear term is at least $48 - 2m$. Therefore $2v \geq (2m - 36) + (48 - 2m) = 12$.

Note that we are in the anticanonical case, and the anticanonical pencil $\mathcal{L}(3; 1^4, [1, 1]^2)$ of \tilde{V}_2 has the (-3) -curve E as a fixed component, and the movable part is the pencil $\mathcal{L}(2; 1^4)$.

We have $b \leq 2m/5 < m$ since $a \leq d - 3m$ and $d/m \geq 16/5$. One has $\mathcal{L}_{V_2} \cdot K_{\tilde{V}_2} = 18m - 6d + a \leq -m$. By Proposition 1.10, it suffices to prove that \mathcal{L}_{V_2} is nef. Let D be an irreducible curve on \tilde{V}_2 such that $\mathcal{L}_{V_2} \cdot D < 0$. Then the proof of Proposition 1.11 shows that D can either be a (-1) -curve or a (-2) -curve or the (-3) -curve E . However E is the curve in the system $\mathcal{L}(1; 0^4, [1, 1]^2)$ and so $\mathcal{L}_{V_2} \cdot E = 6d - 18m - 3a \geq 3m/5 > 0$. By the structure of the anticanonical pencil, the only (-2) -curves on \tilde{V}_2 are G_1, G_2 , and we have $\mathcal{L}_{V_2} \cdot G_i = e$ for $i = 1, 2$. Hence we are reduced to considering the (-1) -curves.

The antibicanonical system $\mathcal{L}(6; 2^4, [2, 2]^2)$ again has E as the fixed part and the movable part is the 4-dimensional system $\mathcal{L}(5; 2^4, [1, 1]^2)$. Since every (-1) -curve is contained in a curve of the antibicanonical system, we see that the (-1) -curves have degree at most 5.

One can take these up in turn, by degree. Those of degree 1 are lines through two of the points; they could be in one of the two systems $\mathcal{L}(1; 1^2, 0^2, [0, 0]^2)$ or $\mathcal{L}(1; 1, 0^3, [1, 0], [0, 0])$. Since $m \geq b$, the first type has the smallest intersection with \mathcal{L}_{V_2} ; this intersection is a , which is positive.

Those of degree 2 are conics through five of the points; the two with smallest intersection with \mathcal{L}_{V_2} are in the systems $\mathcal{L}(2; 1^4, [1, 0], [0, 0])$ and $\mathcal{L}(2; 1^4, [0, 0], [1, 0])$. They meet the system with intersection number $a - (5m - 3c - e)$ which is non-negative by hypothesis.

Those of degree 3 which have smallest intersection with \mathcal{L}_{V_2} are in the linear system $\mathcal{L}(3; 2, 1^3, [1, 1], [1, 0])$ (up to permutations of the points); the intersection number $(9d - 28m - e)/2$ is positive since $d/m \geq 16/5$.

Finally the curve of degree 4 of interest is in the system $\mathcal{L}(4; 2^3, 1, [1, 1]^2)$; this curve has intersection number $6d - 19m$ which is positive since $d/m \geq 16/5$.

Since there are no (-1) -curves of degree 5, we have shown that \mathcal{L} is nef, and this completes the proof. \square

We may find an a satisfying the hypothesis of the previous Lemma if and only if $5m - 3c - e \leq d - 3m$; this is equivalent to $5d - 16m \geq e$, which is true if and only if $d/m \geq 16/5$. This shows that, under the assumptions of this section, condition (i) of Proposition 5.2, namely the non-speciality of the bundles on each component of the degeneration, holds with a in the range of Lemma 6.5.

As for condition (ii) of Proposition 5.2, we set $W_1 = V_2$, $W_2 = V_2 + T_2$, $W_3 = X_0$.

In order to compute $H^0(W_1, \mathcal{L}^{(1)})$ we proceed as in Remark 5.4 since $W_1 = V_2$ is non-normal, and is obtained from the normalization \tilde{V}_2 by identifying the two disjoint (-1) -curves F_1 and F_2 . Applying those arguments, we see that $H^1(W_1, \mathcal{L}^{(1)}) = 0$ will follow from showing that $H^0(\tilde{V}_2, \mathcal{L}_{\tilde{V}_2})$ surjects onto the appropriate quotient of $H^0(F_1, \mathcal{L}_{F_1}) \oplus H^0(F_2, \mathcal{L}_{F_2})$; it therefore suffices to show that the H^0 on \tilde{V}_2 surjects onto the direct sum itself, and this is implied by the following:

Lemma 6.6. *With the same numerical hypotheses as in Lemma 6.5, one has $H^1(\tilde{V}_2, \mathcal{L}_{\tilde{V}_2}(-F_1 - F_2)) = 0$.*

Proof. The argument parallels the proof in Lemma 6.5, and therefore we will be brief. We first show effectivity, and as above it suffices to prove this for $a = 5m - 3c - e$; in this case we compute twice the virtual dimension v to be now

$$2v = (5m - 3c)(45c - 71m) + (39c - 63m) + 6e(31m - 19c - 1) - 4.$$

If $e = 0$, the quadratic part is again at least m , and the linear part is at least $m - c$; since $2m - c > 1$, we have $v \geq 0$. If $e = 1$, the quadratic part is (as above) at least $2m - 36$, and now the linear part is bounded below by $50 - 2m$ so that $2v \geq 4$.

Therefore the bundle is effective. Moreover the intersection with the canonical bundle stays negative. Finally one checks that the only curves that may have negative intersection with the bundle are some of the conics, which may have intersection number -1 at least; this does not cause speciality. \square

Next we proceed to add the surface T_2 , creating W_2 . In this case the double curve is the union of two lines in the plane T_2 , which are glued to G_1 and G_2 on V_2 . The bundle on T_2 , has degree e and the restriction from the plane is surjective. This proves that $H^1(W_1, \mathcal{L}^{(2)}) = 0$.

Finally we add Z_2 , completing the central fibre. This time the double curve is the nodal curve E . We claim that the restriction map from the sections of the bundle on Z_2 to the sections on E is surjective, which is sufficient for the criterion of Proposition 5.2; this then will give us the desired non-speciality of the bundle on X_0 . In fact, as we saw, the bundle on Z_2 is Cremona equivalent to $\mathcal{L}(4\alpha - 3a; (\alpha - a)^6)$ and E turns out to be Cremona equivalent to a nodal curve in the linear system $\mathcal{L}(3; 1^6)$. The difference linear system is $\mathcal{L}(4\alpha - 3a - 3; (\alpha - a - 1)^6)$. This system is still excellent and therefore non-special by Proposition 1.6. This implies the surjectivity of the restriction map.

We have now checked all the necessary details to conclude the following:

Corollary 6.7. *If $16/5 \leq d/m < 10/3$ then the system $\mathcal{L}(d; m^{10})$ has the expected dimension.*

7 The third degeneration: throwing the conics

In what follows we assume that $19/6 \leq d/m < 16/5$. Write d , b and α as in (6.1), (6.2) and (6.3). In this section we will assume the inequality $b > 2a$; this is equivalent to $a < 5m - 3c - e$. We will also often assume $a \leq \alpha + 1$.

We consider the second degeneration, and we additionally twist the line bundle on the threefold by $-(b - 2a - e)T_2 = -(5m - a - 3c - 2e)T_2$.

We then have the configuration of the second degeneration, but the bundles on V_2 and T_2 have now changed. Since T_2 does not meet Z_2 , the bundle on Z_2 is unchanged; it is still of the form $\mathcal{L}(10\alpha - 6a; 6\alpha - 3a, (3\alpha - 2a)^6)$, which, as we saw in (6.4), is Cremona equivalent to $\mathcal{L}(4\alpha - 3a; (\alpha - a)^6)$.

The bundle on the plane T_2 now has degree $2b - 4a - e > 0$.

The bundle on V_2 , which used to pull-back on \tilde{V}_2 as $\mathcal{L}(2m + a; m^4, [b, b - e]^2)$, has been twisted by $-(b - 2a - e)(G_1 + G_2)$, and now pulls back to $\mathcal{L}(2m + a; m^4, [2b - 2a - e, 2a]^2)$.

Let us consider the proper transforms on \tilde{V}_2 of the two conics C_1, C_2 which are in the systems

$$\mathcal{L}(2; 1^4, [1, 0], [0, 0]), \quad \mathcal{L}(2; 1^4, [0, 0], [1, 0]).$$

Their intersection number with the system on \tilde{V}_2 is $-(2(b - 2a) - e)$, which is negative.

In this third degeneration we now execute two 2-throws, one for each C_i . Each C_i will be removed $2b - 4a - e$ times from the system on V_2 , and then blown down. Each C_i meets the double curve twice, but on different components: C_i meets G_i transversally, and also meets the curve E . Therefore we will create four new $[b - 2a, b - 2a - e]$ -points, two on the blow-up of Z_2 (let us call that surface Z_3 now), and two on the blow-up of T_2 (which we call T_3 now). The surfaces Z_3 and T_3 will now meet along two curves A_1, A_2 , the exceptional divisors of the second blow-ups at the two points.

We also create two new planes $U_{1,3}$ and $U_{2,3}$ from the 2-throw construction with C_1 and C_2 , respectively. They will meet both T_3 and Z_3 along lines, at the first blow-ups of each infinitely near point. The bundle, restricted to $U_{i,3}$, has degree e .

The surface V_2 has both conics blown down; we call the result V_3 (and its normalization \tilde{V}_3). The linear system on \tilde{V}_3 is that of \tilde{V}_2 , with the two conics removed $2b - 4a - e$ times. It will be convenient to set

$$\mu = 6d - 19m$$

which is non-negative. Then linear system on \tilde{V}_3 corresponds to the system $\mathcal{L}(9a + 2\mu; (4a + \mu)^4, [2a, 2a]^2)$.

Thus, this third degeneration consists of the five surfaces $V_3, Z_3, T_3, U_{1,3}$, and $U_{2,3}$, with double curves as shown in Figure 6.

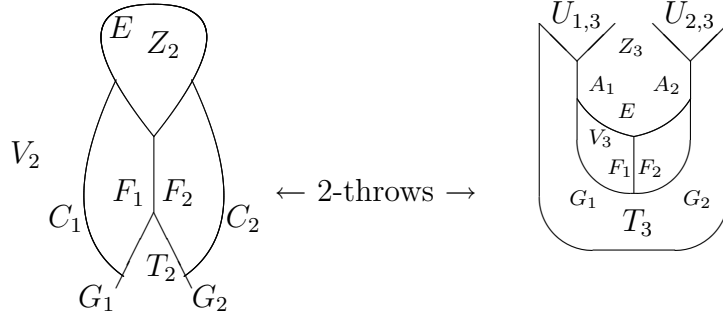


Figure 6: throwing the conics C_1, C_2

Lemma 7.1. *If $d/m \geq 19/6$ and $a \geq 0$ then the system $\mathcal{L}_{\tilde{V}_3}$ is non-empty and non-special.*

Proof. The system on \tilde{V}_3 is not standard, and we have the following series of quadratic transformations:

$$\begin{array}{ccccccc} 9a + 2\mu; & 4a + \mu, & 4a + \mu, & 4a + \mu, & 4a + \mu, & [2a, 2a], & [2a, 2a] \\ 6a + \mu; & a, & a, & a, & 4a + \mu, & [2a, 2a], & [2a, 2a] \\ 4a + \mu; & a, & a, & a, & 2a + \mu, & 2a, & 2a \\ 2a + \mu; & a, & a, & a & \mu & & \end{array}$$

If $a \leq \mu$ the final system is excellent. Otherwise, perform a quadratic transformation based at the three points of multiplicity a , getting the system $\mathcal{L}(3a; \mu^4)$, which is excellent. \square

The bundle \mathcal{L}_{Z_3} on Z_3 is Cremona equivalent to $\mathcal{L}(4\alpha - 3a; (\alpha - a)^6, [b - 2a, b - 2a - e]^2)$, where the two compound multiple points lie on an irreducible nodal cubic curve passing through the six points of multiplicity $\alpha - a$; this cubic is the image of the original double curve E under the Cremona transformation. The existence of E implies that we are in the strong anticanonical case.

Lemma 7.2. *If $19/6 \leq d/m < 16/5$ and $0 \leq a \leq \alpha + 1$ then the system \mathcal{L}_{Z_3} is non-empty and non-special.*

Proof. We have

$$\mathcal{L}_{Z_3} \cdot K_{Z_3} = -2\mu - a \tag{7.3}$$

which is non-positive. Moreover $b - 2a > \alpha - a$, since this is equivalent to $16m \geq 5d + e$. Therefore the three largest multiplicities of the system $\mathcal{L}(4\alpha - 3a; (\alpha - a)^6, [b - 2a, b - 2a - e]^2)$ are those of the compound points. Hence, after removing six innocuous (-1) -curves in case $a = \alpha + 1$, this system is standard if $3b - e - 3a \leq 4\alpha$, which is equivalent to $d/m \geq 54/17$.

Note that $19/6 < 54/17 < 16/5$. Thus if $d/m \geq 54/17$, we also have $\mu > 0$ and the system is excellent; therefore by part (iii) of Proposition 1.6, the system is non-special.

If $d/m < 54/17$, we perform a quadratic transformation based at the three points of largest multiplicity (the $b - 2a$, $b - 2a$, and $b - 2a - e$ points), obtaining the system $\mathcal{L}(25c + 12e - 39m - 3a; (\alpha - a)^6, 7d - 22m - a, 7d - 22m - a - e, 7d - 22m - a, 5m - 3c - a - 2e)$. Now the three largest multiplicities are $7d - 22m - a$, $7d - 22m - a$, and $5m - 3c - a - 2e$ (since $d/m \geq 19/6$); their sum is equal to the degree $25c + 12e - 39m - 3a$, and so this system is standard. We still have the intersection with the canonical class given by (7.3) and so, if either $d/m > 19/6$ or $a > 0$, this system is excellent; we conclude by applying part (iii) of Proposition 1.6 as above.

If $d/m = 19/6$ and $a = 0$, then the system is only almost excellent. Its restriction to the anticanonical curve E has degree 0, and, because of the generality of the original choice of the points, the restriction to E is a non-trivial bundle, which therefore has no H^1 , since E has arithmetic genus 1. The kernel of the restriction to E is excellent, and therefore is non-special by part (iii) of Proposition 1.6. The usual restriction exact sequence now shows that the original system \mathcal{L}_{Z_3} is non-special as well. \square

Let us now turn our attention to the surface T_3 , the quadruple blow-up of the plane T_2 ; the linear system there is $\mathcal{L}_{T_3} = \mathcal{L}(2b - 4a - e; [b - 2a, b - 2a - e]^2)$.

Lemma 7.4. *The system \mathcal{L}_{T_3} is non-empty and non-special.*

Proof. The linear system \mathcal{L}_{T_3} is composed of $b - 2a - e$ conics in a pencil of conics bitangent to the pair of lines corresponding to G_1 and G_2 , plus, if $e = 1$, the fixed line through the base points. This system is clearly non-special. \square

Finally the surfaces $U_{i,3}$ are planes, with systems of non-negative degree on them, hence also non-special.

We now turn our attention to check the required surjectivity criteria from Proposition 5.2. In this case we set

$$W_1 = V_3, W_2 = W_1 + T_3, W_3 = W_2 + Z_3, W_4 = X_0.$$

The analysis is similar to that for the second degeneration. We will need the following two lemmas.

Lemma 7.5. *If $19/6 < d/m < 16/5$, and $a \geq 0$ then $H^1(\tilde{V}_3, \mathcal{L}_{\tilde{V}_3}(-F_1 - F_2)) = 0$.*

Proof. As we saw in the proof of Lemma 7.1, the system $\mathcal{L}_{\tilde{V}_3}$ corresponds to $\mathcal{L}(9a + 2\mu; (4a + \mu)^4, [2a, 2a]^2)$. By subtracting F_1 and F_2 , one sees that the corresponding system is $\mathcal{L}(9a + 2\mu; (4a + \mu)^4, [2a + 1, 2a]^2)$. If $a = 0$, this is the system $\mathcal{L}(2\mu; \mu^4, 1^2)$ which is non-special since $\mu > 0$. Assume $a > 0$. Proceeding as in the proof of Lemma 7.1, we see that two (-1) -curves split once and the residual system is Cremona equivalent to $\mathcal{L}(2a + \mu - 2; a^3, \mu - 2)$.

If $\mu = 1$ one more (-1) -curve splits once and the residual system is $\mathcal{L}(2a - 1; a^3)$. If $a = 1$, this is empty with no H^1 . If $a > 1$, three more (-1) -curves split once and the residual system is $\mathcal{L}(2a - 4; (a - 2)^3)$, which is non-special.

If $\mu > 1$, and $a \leq \mu - 2$, the system is excellent, hence non-special. Otherwise, perform a Cremona transformation at the three points of multiplicity a , thus obtaining the system $\mathcal{L}(3a; (\mu - 2)^4)$, which is excellent, hence non special. \square

Recall that the double curve created by adding Z_3 to W_2 is $E + A_1 + A_2$, where the A_i 's are the exceptional divisors of the second blow-ups of the two compound multiple points on Z_3 . (As shown in Figure 6, these are the intersection curves of Z_3 with T_3 .)

Lemma 7.6. *If $19/6 < d/m < 16/5$, and $0 \leq a \leq \alpha+1$ then $H^1(Z_3, \mathcal{L}_{Z_3}(-E-A_1-A_2)) = 0$.*

Proof. The proof parallels the one of Lemma 7.2.

Proceeding as in (6.4), we see that the system in question corresponds to $\mathcal{L}(4\alpha - 3a - 3; (\alpha - a - 1)^6, [b - 2a - e, b - 2a - 1]^2)$, whose intersection with the canonical bundle is $1 - 2\mu - 5a$.

One has $b - 2a - e > \alpha - a - 1$, since this is equivalent to $16m - 5d - e \geq 0$. If $3b - 2e - 1 - 6a \leq 4\alpha - 3a - 3$, which is equivalent to $17c + 9e - 27m \geq 2$, the system is excellent. Otherwise perform a quadratic transformation based at the points of largest multiplicities, i.e. $b - 2a - e, b - 2a - e, b - 2a - 1$. The resulting system is $\mathcal{L}(25c + 13e - 39m - 3a; (\alpha - a - 1)^6, 14 + 7e - 22m - a - 2, 14c + 8e - 22m - a - 3, 14c + 7e - 22m - 2, 5m - 3c - a - e - 1)$. Since $\mu > 0$, the largest multiplicities are $14c + 7e - 22m - a - 2$ and $5m - 3c - a - e - 1$ and the system is excellent. \square

Our next non-speciality result is as follows.

Proposition 7.7. *If $19/6 < d/m < 16/5$ then the system $\mathcal{L}(d; m^{10})$ has the expected dimension.*

Proof. The non-speciality of the system on W_1 follows from Lemma 7.5 for any $a \geq 0$.

As for the glueing of T_3 to create W_2 , we note that the double curve is $G = G_1 + G_2$, which is the proper transform on T_3 of two lines in the plane T_2 . The bundle on T_3 restricts to G as the trivial bundle, and surjectivity follows by remarking that G is not in the base locus of the linear system on T_3 (see the proof of Lemma 7.4).

If $0 \leq a \leq \alpha+1$, Lemma 7.2 implies that we have non-speciality on W_3 . Finally attaching the two planes $U_{1,3}, U_{2,3}$ does not create any speciality, since the bundles there have degree e , and we have non-speciality on the central fibre W_4 . \square

8 Generality and transversality

In this section we focus on the generality of our choices in the above constructions in order to prove some transversality properties needed in the sequel.

8.1 Correspondence generality.

First, let us go back to the second degeneration. Consider the two curves F_1 and F_2 on V_1 which are identified via a projective transformation $\omega : F_1 \rightarrow F_2$ in order to get V_2 . Let p_i, q_i the intersection points of F_1 with E and G_i , $i = 1, 2$, respectively. One has $\omega(p_1) = p_2$ and $\omega(q_1) = q_2$. Our first claim is that ω can be assumed to be general, given this constraint. This gives a one dimensional family of such projective transformations, depending on a parameter varying in \mathbb{C}^* .

To see this, go back to the first degeneration. After we choose the six general points on \mathbb{F} , we have the cubic curve C on \mathbb{F} , which we will throw in the second degeneration. This curve cuts E in two points x_1, x_2 , which will be blown up twice on \mathbb{P} in the second degeneration creating the exceptional curves F_1, G_1 and F_2, G_2 . Consider the projective transformations of \mathbb{P} fixing x_1 and x_2 . They form a group Ω of dimension 4. This group acts also on the double blow up of \mathbb{P} at x_1 and x_2 , and therefore it acts on all the curves F_1, G_1 and F_2, G_2 . One sees that Ω induces on F_1, F_2 the full group of pairs of projective transformations fixing p_1, q_1 and p_2, q_2 . We leave the easy proof to the reader.

Note now that we can act by Ω on our choices of the remaining four general points on \mathbb{P} , that we blow up creating V_1 . This implies our claim about the generality of ω .

The same considerations work also for the two curves A_1, A_2 on Z_3 in the third degeneration, and the projective transformation between them induced by the pencil of conics on the plane T_2 bitangent to the pair of lines corresponding to G_1 and G_2 .

8.2 Configuration generality.

Next we go back to the third degeneration.

As we saw in the proof of Lemma 7.1, the system $\mathcal{L}_{\tilde{V}_3}$ is Cremona equivalent to the system $\mathcal{L}(2a + \mu; a^3, \mu)$. The reader may verify that, under the series of quadratic transformations performed in the proof of Lemma 7.1, the two curves F_1, F_2 map to two lines L_1, L_2 passing through the point of multiplicity μ . This point arises as the contraction of the quartic curve $\mathcal{L}(4; 2, 2, 2, 1, [1, 1]^2)$. The remaining three base points y_1, y_2, y_3 of multiplicity a are general, and the lines joining them correspond to the quartic curves $\mathcal{L}(4; 2, 2, 1, 2, [1, 1]^2)$, $\mathcal{L}(4; 2, 1, 2, 2, [1, 1]^2)$, $\mathcal{L}(4; 1, 2, 2, 2, [1, 1]^2)$.

If we perform a further quadratic transformation based at y_1, y_2, y_3 , then L_1, L_2 are mapped to two conics Γ_1, Γ_2 , intersecting at four points a_1, \dots, a_4 which arise as the contractions of all the aforementioned quartic curves. The above generality considerations, tell us that a_1, \dots, a_4 correspond to general points of F_1, F_2 .

In the resulting final Cremona transformation, the curve E maps to a line L meeting Γ_i at two points u_i, v_i , where u_i is the image of the point p_i intersection of F_i with E , and v_i is the image of the intersection of F_i with G_i . Note that in fact the final Cremona transformation contracts the two reducible (-1) -cycles on V_2 formed by $C_i + G_i$ to points on L which are exactly the points $v_i, i = 1, 2$.

The configuration formed by $\Gamma = \Gamma_1 + \Gamma_2$ and L is completely general, because making the inverse Cremona transformation we arrive at the system $\mathcal{L}_{\tilde{V}_3}$, whose base points are general with the only constraint of the two infinitely near ones.

Finally, let us remark that there is a one parameter family \mathcal{F} of projective transformations of the plane mapping u_1 to u_2, v_1 to v_2 and Γ_1 to Γ_2 . The induced transformations between Γ_1 and Γ_2 correspond to the projective transformations between F_1 and F_2 mapping p_1 to p_2 and q_1 to q_2 . Let $\omega \in \mathcal{F}$ be general. The points $s_i = \omega(a_i)$ lie on Γ_2 and the points $t_i = \omega^{-1}(a_i)$ on $\Gamma_1, i = 1, \dots, 4$. On the whole we have eight points $\{s_i, t_i\}_{1 \leq i \leq 4}$ forming a divisor D_ω on the curve Γ .

Lemma 8.1. *In the above setting, for general $\omega \in \mathcal{F}$, neither D_ω is cut out on Γ by a conic nor $2D_\omega$ is cut out on Γ by a quartic.*

Proof. As for the first assertion, suppose that for the general $\omega \in \mathcal{F}$ there is conic Γ_ω cutting out D_ω on Γ . For some special ω , one of the s_i points, say s_1 , coincides with one of the points a_1, \dots, a_4 , whereas s_2, s_3, s_4 do not lie on Γ_1 . By the generality assumption, for this ω the points t_1, \dots, t_4 stay distinct from a_1, \dots, a_4 . Then the conic Γ_ω must coincide with Γ_1 , since it has five points in common with it. But then it does not contain s_2, s_3, s_4 which is a contradiction. The proof of the second assertion is similar and can be left to the reader. \square

8.3 Transversality.

Still referring to the second, or third, degeneration, consider the surface V_2 , or V_3 , which we will denote by V here, and its normalization \tilde{V} . On \tilde{V} we have the two curves F_1, F_2 which are glued via the correspondence ω to form V . Consider the linear system $\mathcal{L}_{\tilde{V}}$, and let r be the dimension of the linear series \mathcal{R}_F it cuts out on $F = F_1 + F_2$, and $d = \mathcal{L} \cdot F_i$, $i = 1, 2$. We denote by \mathcal{R}_{F_i} the linear series cut out by $\mathcal{L}_{\tilde{V}}$ on F_i , $i = 1, 2$, which both have degree d and dimension r . Then we can consider following two subvarieties of $\text{Sym}^d(F_1) \times \text{Sym}^d(F_2)$:

- X , of dimension r , consisting of all pairs (D_1, D_2) such that $D_1 + D_2 \in \mathcal{L}_F$;
- Y , of dimension d , consisting of all pairs (D_1, D_2) such that $D_2 = \omega(D_1)$.

One has the following transversality statement:

Lemma 8.2. *In the above setting, for general choices, X and Y intersect properly inside $\text{Sym}^d(F_1) \times \text{Sym}^d(F_2)$, i.e. $\dim(X \cup Y) = \max\{r - d, -1\}$, unless either the point p_i , or the point q_i is inflectional for \mathcal{R}_{F_i} , for both $i = 1, 2$.*

Proof. The proof is the same as the one of Proposition 3.1 of [3] and therefore we do not dwell on it here. \square

A similar lemma holds for A_1 and A_2 on Z .

9 The fourth degeneration: throwing the quartics

Now we want to analyze the situation when the ratio d/m is at most $19/6$. We perform the same 2-throws, up through the third degeneration, and then make our fourth degeneration by throwing four curves in V_3 corresponding to certain quartics.

In what follows we assume that $174/55 \leq d/m \leq 19/6$. We will use the same notation as above. In particular $\alpha = d - 3m$. We will additionally set

$$\ell = -\mu = 19m - 6d$$

which is non-negative and

$$\ell = 2r - s, \quad \text{with } s \in \{0, 1\}.$$

Note that

$$d = 3\ell + 19\alpha, \quad m = \ell + 6\alpha$$

and $d/m \geq 174/55$ is equivalent to

$$\alpha \geq 9\ell.$$

Recall that in the third degeneration we have five surfaces:

- V_3 , with linear system of the form $\mathcal{L}(9a - 2\ell; (4a - \ell)^4, [2a, 2a]^2)$;
- Z_3 , with linear system of the form $\mathcal{L}(10\alpha - 6a; 6\alpha - 3a, (3\alpha - 2a)^6, [b - 2a, b - 2a - e]^2)$;
- T_3 , with linear system of the form $\mathcal{L}(10m - 3d - 2a; [b - 2a, b - 2a - e]^2)$;
- $U_{1,3}$ and $U_{2,3}$, planes, with linear systems of degree e .

We now note that each of the four disjoint curves on V_3 corresponding to the quartics

$$\mathcal{L}(4; 2, 2, 2, 1, [1, 1]^2), \quad \mathcal{L}(4; 2, 2, 1, 2, [1, 1]^2), \quad \mathcal{L}(4; 2, 1, 2, 2, [1, 1]^2), \quad \mathcal{L}(4; 1, 2, 2, 2, [1, 1]^2)$$

we already met in §8.2 has intersection number $-\ell$ with the system on V_3 . Each meets the double curve in two points, along F_1 and F_2 , which the reader will recall are identified (giving the self-double curve of V_3) in the second degeneration.

We perform a 2-throw for each of them, resulting in our fourth degeneration. This will consist of nine surfaces:

- V_4 , the transform of V_3 . The normalization \tilde{V}_4 of V_4 is obtained from \tilde{V}_3 , with an additional eight double blow-ups, four each on the curves F_i , corresponding to eight $[r, r - s]$ -points, and by blowing down the curves corresponding to the four quartics. The four 2-throws results in removing each quartic ℓ times from the bundle on V_4 , which therefore corresponds to a linear system of the form $\mathcal{L}(9a - 18\ell, (4a - 8\ell)^4, [2a - 4\ell, 2a - 4\ell]^2, [r, r - s]^8)$;
- $Z_4, T_4, U_{1,4}$, and $U_{2,4}$, unchanged from the corresponding surfaces in the third degeneration, with the same bundles;
- four new planes $Y_i, i = 1, \dots, 4$, with bundles of degree s on them; these planes arise from the four 2-throws.

The picture of the central fibre of the fourth degeneration is shown in Figure 7.

Note that, in order for the system on V_4 to be non-empty it is necessary that $a \geq 2\ell$.

Lemma 9.1. *If $a > 4\ell + 1$ then the system on V_4 is non-empty and non-special. If $\ell \leq 2$ the same holds if $a \geq 4\ell + 1$.*

Proof. For the system on V_4 , we have the following series of quadratic transformations which do not involve the eight $[r, r - s]$ -points:

$$\begin{array}{ccccccc} 9(a - 2\ell); & \frac{4(a - 2\ell)}{a - 2\ell}, & \frac{4(a - 2\ell)}{a - 2\ell}, & \frac{4(a - 2\ell)}{a - 2\ell}, & 4(a - 2\ell), & [2(a - 2\ell), 2(a - 2\ell)], & [2(a - 2\ell), 2(a - 2\ell)] \\ 6(a - 2\ell); & \frac{4(a - 2\ell)}{a - 2\ell}, & \frac{4(a - 2\ell)}{a - 2\ell}, & \frac{4(a - 2\ell)}{a - 2\ell}, & \frac{4(a - 2\ell)}{a - 2\ell}, & [2(a - 2\ell), 2(a - 2\ell)], & [2(a - 2\ell), 2(a - 2\ell)] \\ 4(a - 2\ell); & a - 2\ell, & a - 2\ell, & a - 2\ell, & \frac{2(a - 2\ell)}{a - 2\ell}, & \frac{2(a - 2\ell)}{a - 2\ell}, & \frac{2(a - 2\ell)}{a - 2\ell} \\ 2(a - 2\ell); & \frac{a - 2\ell}{a - 2\ell}, & \frac{a - 2\ell}{a - 2\ell}, & \frac{a - 2\ell}{a - 2\ell} & & & \\ a - 2\ell & & & & & & \end{array}$$

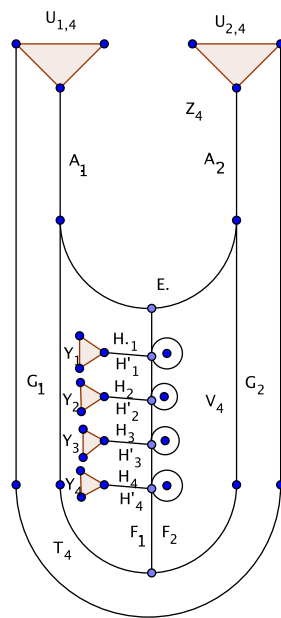


Figure 7: the central fibre of the fourth degeneration

Hence the system on V_4 Cremona reduces to $\mathcal{L}(a - 2\ell; [r, r - s]^8)$. We will therefore prove that this system is non-empty and non-special, by working on the plane blown up at the eight compound points rather than on \tilde{V}_4 .

As we saw in §8.2, under the previous series of quadratic transformations, the two curves F_i map to two conics Γ_i , $i = 1, 2$. Each of them contains four of the $[r, r - s]$ -points, i.e. the points $t_i \in \Gamma_1$ and $s_i \in \Gamma_2$, $i = 1, \dots, 4$. The divisor D of degree 8 on Γ consisting of all these points is not in the bicanonical series of Γ by Lemma 8.1.

The curve $\Gamma = \Gamma_1 + \Gamma_2$ is 1-connected, with arithmetic genus three. Restricting to Γ , we have for each $i = 0, \dots, r - 1$ the exact sequences

$$\begin{aligned} 0 \rightarrow \mathcal{L}(a - 2\ell - 4(i + 1); [r - i - 1, r - s - i - 1]^8) \rightarrow \mathcal{L}(a - 2\ell - 4i; [r - i, r - s - i]^8) \\ \rightarrow \mathcal{L}(a - 2\ell - 4i; [r - i, r - s - i]^8)_{|\Gamma} \rightarrow 0 \end{aligned}$$

where, by abusing notation, we use the same symbol to denote a linear system and the corresponding line bundle.

One has

$$\mathcal{L}(a - 2\ell - 4i; [r - i, r - s - i]^8) \cdot \Gamma = 4(a - 4\ell) > 4$$

if $a > 4\ell + 1$, in which case the restrictions of all the above line bundles to Γ have no H^1 . Therefore, the middle linear system is non-empty and non-special if the system on the left is non-empty and non-special. Since the system with $i = r - 2$ is $\mathcal{L}(a - 4\ell - 2s + 4; [1, 1 - s]^8)$, which is non-empty and non-special, we conclude by induction.

Let now $a = 4\ell + 1$. If $\ell = 0$ the system is clearly non-special. If $1 \leq \ell \leq 2$, then $r = 1$ and $i = 0$. Then $\mathcal{L}(2\ell + 1; [r, r - s]^8)_{|\Gamma}$ has degree 4. However, by Lemma 8.1, $\mathcal{L}(2\ell + 1; [r, r - s]^8)_{|\Gamma}$ is still not special and the same argument as above can be applied to conclude. \square

Next we deal with the system on Z_4 , of the form $\mathcal{L}(10\alpha - 6a; 6\alpha - 3a, (3\alpha - 2a)^6, [b - 2a, b - 2a - e]^2)$, which is the same as the one on Z_3 . We saw in (6.4) that this system is Cremona equivalent to $\mathcal{L}(4\alpha - 3a; (\alpha - a)^6, [b - 2a, b - 2a - e]^2)$ and the ten multiple points all lie on the irreducible nodal cubic curve E . Hence we are in the strong anticanonical case.

We first note that $4\ell < \alpha - 2\ell \leq (69d - 218m)/2$. The first inequality is equivalent to $d/m > 117/37$ and we have $174/55 > 117/37$. The second inequality is equivalent to $d/m \geq 136/43$ and we have also $174/55 > 136/43$.

Lemma 9.2. *If $174/55 \leq d/m \leq 19/6$, and $4\ell < a \leq \alpha - 2\ell$, then the linear system on Z_4 is non-empty and non-special.*

Proof. One has

$$\mathcal{L}_{Z_4} \cdot K_{Z_4} = 2\ell - a < 0.$$

Next we note that $b - 2a > \alpha - a$, since this is equivalent to $d/m < 16/5$. Hence the three largest multiplicities of the system on Z_4 are $b - 2a$, $b - 2a$, and $b - 2a - e$. Comparing their sum $3b - 6a - e$ with the degree $4\alpha - 3a$, we see that $3b - 6a - e > 4\alpha - 3a$, since this is equivalent to $d/m < 54/17$, and $19/6 < 54/17$. Hence the system is not standard, and we may apply a quadratic transformation based at these three points. This results in the system

$$\mathcal{L}(8\alpha - 3b + e; (\alpha - a)^6, b - 2a - e, (4\alpha + a - 2b + e)^2, 4\alpha + a - 2b)$$

where we have suppressed the infinitely near nature of the points in the notation. Now we have that $\alpha - a \geq 4\alpha + a - 2b + e$ since this is equivalent to $d/m \leq 19/6$. So in this system, the three largest multiplicities are $b - 2a - e$, $\alpha - a$, and $\alpha - a$. Their sum is at least the degree, since $2\alpha + b - 4a - e \geq 8\alpha - 3b + e$ is equivalent to $d/m \leq 19/6$. Perform another quadratic transformation centered at these three points; we obtain the system

$$\mathcal{L}(14\alpha - 7b + 4a + 3e; (\alpha - a)^4, 6\alpha + 2a - 3b + e, (7\alpha + 3a - 4b + 2e)^2, (4\alpha + a - 2b + e)^2, 4\alpha + a - 2b).$$

The three multiplicities $\alpha - a$, $\alpha - a$, and $6\alpha + 2a - 3b + e$ have sum at least the degree since $d/m \leq 19/6$; another quadratic transformation results in the system

$$\mathcal{L}(20\alpha - 11b + 8a + 5e; (\alpha - a)^2, 12\alpha - 7b + 6a + 3e, (7\alpha + 3a - 4b + 2e)^4, (4\alpha + a - 2b + e)^2, 4\alpha + a - 2b).$$

Again the three multiplicities $\alpha - a$, $\alpha - a$, and $12\alpha - 7b + 6a + 3e$ have sum at least the degree since $d/m \leq 19/6$; one further quadratic transformation gives us the system

$$\mathcal{L}(26\alpha - 15b + 12a + 7e; (7\alpha + 3a - 4b + 2e)^6, (4\alpha + a - 2b + e)^2, 4\alpha + a - 2b, 18\alpha - 11b + 10a + 5e).$$

The multiplicity $18\alpha - 11b + 10a + 5e$ is non-negative, since this is equivalent to $2a \leq 69d - 218m$.

Note that

$$\begin{aligned} 26\alpha - 15b + 12a + 7e &= 48d - 153m - 3a + c = c - m - 3a - 8\ell \\ 18\alpha - 11b + 10a + 5e &= 34d - 109m - a + c = c - a - m - 2\alpha - 6\ell \\ 7\alpha + 3a - 4b + 2e &= 13d - 41m - a = \alpha - 2\ell - a \\ 4\alpha + a - 2b + e &= 7d - 22m + a = \alpha - \ell - a, \end{aligned}$$

We now observe that if $d/m \geq 174/55$, then $c - a - m - 2\alpha - 6\ell \geq \alpha - \ell - a$ and the three largest multiplicities here are

$$18\alpha - 11b + 10a + 5e, 4\alpha + a - 2b + e, 4\alpha + a - 2b + e;$$

since their sum is $c - a - m - 2\alpha - 6\ell + 2(\alpha - \ell - a) = c - m - 3a - 8\ell$ which is the degree of the system, the system is standard, and excellent, and we are done. \square

Note that, in the above proof, if $d/m < 174/55$, then the system is not standard.

The system on T_4 is the same as that on T_3 , so we have the same criterion as in Lemma, i.e. we have the system non-empty and non-special if $10m - 3d \geq 2a$.

Lemma 9.3. *If $d/m \leq 19/6$ and $a \leq \alpha - 2\ell$ then the system on T_4 is non-empty and non-special.*

Proof. We must show that the hypothesis implies $2a \leq 10m - 3d$, which will be the case if $2\alpha - 4\ell \leq 10m - 3d$. This is equivalent to $d/m \leq 92/29$, which is true, since $19/6 < 92/29$. \square

We deal now with checking the surjectivity criteria from Proposition 5.2. In this case we set

$$W_1 = V_4, W_2 = W_1 + Z_4, W_3 = W_2 + T_4, W_4 = W_3 + U_{1,4} + U_{2,4}, W_5 = X_0.$$

Since W_1 is again non-normal, we have to deal with the self-double curves. We abuse notation, and still denote by F_i , $i = 1, 2$, and E the proper transform on \tilde{V}_4 of these curves on \tilde{V}_3 . Let H_1, \dots, H_4 be the second exceptional divisors of four of the second sets of eight compound singularities, the ones that meet F_1 . Let $D = F_1 + H_1 + \dots + H_4$. With this notation, V_4 is obtained by suitably gluing D with its counterpart D' formed by the proper transform of F_2 and the four curves H'_1, \dots, H'_4 analogous to H_1, \dots, H_4 .

Lemma 9.4. *Let $174/55 \leq d/m \leq 19/6$. Then:*

- (i) *if $a > 4\ell + 3$, then $H^1(\tilde{V}_4, \mathcal{L}_{\tilde{V}_4}(-D - D')) = 0$. If $\ell = 2$ the same conclusion holds if $a \geq 11$. If $0 \leq \ell \leq 1$ the same conclusion holds for all a ;*
- (ii) *if $a > 4\ell + 4$, then $H^1(\tilde{V}_4, \mathcal{L}_{\tilde{V}_4}(-D - D' - E)) = 0$. If $\ell = 2$ the same conclusion holds if $a \geq 12$. If $0 \leq \ell \leq 1$ the same conclusion holds for all a .*

Proof. To prove (i), one checks that the system $\mathcal{L}_{\tilde{V}_4}(-D - D')$ corresponds to $\mathcal{L}(9(a - 2\ell); (4(a - 2\ell))^4, [2(a - 2\ell) + 1, 2(a - 2\ell)]^2, [r - s, r - 1]^8)$. There are six (-1) -curves which intersect this system negatively, namely the four quartics we threw in this section and the two conics we threw in the third degeneration. The intersection number of the system with the conics is -1 , whereas the intersection with the quartics is -2 . The conics therefore split once and the quartics twice. However all these (-1) -curves have been contracted on \tilde{V}_4 , so $\mathcal{L}_{\tilde{V}_4}(-D - D')$ actually corresponds to the system obtained from $\mathcal{L}(9(a - 2\ell); (4(a - 2\ell))^4, [2(a - 2\ell) + 1, 2(a - 2\ell)]^2, [r - s, r - 1]^8)$ by removing these (-1) -curves which appear in its fixed part. This residual system is Cremona equivalent to $\mathcal{L}(a - 2\ell - 4; [r - s, r - 1]^8)$. The case $0 \leq \ell \leq 1$ is obvious. Otherwise, if $a > 4\ell + 3$ we note that the intersection with the curve Γ which we considered in the proof of Lemma 9.1 is $4(a - 4\ell - 2) > 4$ and the argument of that Lemma applies directly to conclude the proof of the first assertion. If $\ell = 2$ the intersection with the curve Γ has degree 4, but it is not special by Lemma 8.1 and we finish as before.

As for (ii), note that, as above, $\mathcal{L}_{\tilde{V}_4}(-D - D' - E)$ corresponds to a system which is Cremona equivalent to $\mathcal{L}(a - 2\ell - 5; [r - s, r - 1]^8)$. Again, the case $0 \leq \ell \leq 1$ is clear. Otherwise note that the intersection with the curve Γ is $4(a - 4\ell - 3) > 4$ and the argument of Lemma 9.1 applies again to conclude the proof in this case too. For $\ell = 2$ the same argument as above applies. \square

Going back now to the surface $W_1 = V_4$, we want to apply the criterion for non-speciality as in Remark 5.4. If C is the double curve of V_4 , then its pull-back on \tilde{V}_4 is the reducible curve $D + D'$. The surjectivity $H^0(\tilde{V}_4, \mathcal{L}_{\tilde{V}_4}) \rightarrow H^0(D + D', \mathcal{L}_{D + D'})$ is a consequence of Lemma 9.4, if either $a > 4\ell + 3$ or $a \geq 11$ and $\ell = 2$ or $\ell \leq 1$. The double curve C is gotten from D , or D' , by gluing four pairs of points, hence C has arithmetic genus 4. Since \mathcal{L}_D has degree at least 8, then \mathcal{L}_C is non special. Thus the criterion in Remark 5.4 can be applied and the system is non-special if either $a > 4\ell + 3$ or $a \geq 11$ and $\ell = 2$ or $\ell \leq 1$.

Now, in passing from W_1 to W_2 we glue the surface Z_4 to W_1 , along the proper transforms of the curve E . We want to apply the criterion for non-speciality given by Proposition 5.2.

Condition (i) of that criterion is verified by Lemmas 9.1 and 9.2 as soon as $a \geq 4\ell + 2$. Condition (ii) is a consequence of $H^1(V_4, \mathcal{L}_{V_4}(-E)) = 0$. An obvious variation of Remark 5.4 shows that this is in turn a consequence of $H^1(\tilde{V}_4, \mathcal{L}_{\tilde{V}_4}(-D - D' - E)) = 0$, which we have as soon as either $a > 4\ell + 4$, or $a \geq 12$ and $\ell = 2$, or $\ell \leq 1$.

Therefore, if either $a > 4\ell + 4$ or $a \geq 12$ and $\ell = 2$, or $\ell \leq 1$ and $a \geq 4\ell + 1$ also the system on W_2 is non-special. Furthermore the restriction map $H^0(W_2, \mathcal{L}) \rightarrow H^0(Z_4, \mathcal{L}_{Z_4})$ is surjective.

Next we glue T_4 , creating W_3 , and to do this we will choose a specific value of a . The double curve is $\Delta = A_1 + A_2 + G_1 + G_2$. Again we apply Remark 5.4.

The surjectivity there is implied by the following:

Lemma 9.5. *Assume $a > 2\ell - 2$, $174/55 \leq d/m \leq 19/6$ and take $a = \alpha - 2\ell - h$, with:*

(i) $h = 1$ if either:

(a) $\ell \geq 4$ or

(b) $\ell \leq 3$ and $\alpha \geq 7\ell + 7$;

(ii) $h = 2$ if either:

(a') $\ell = 3$ and $\alpha = 27$ or

(b') $\ell \leq 2$ and $7\ell + 5 \leq \alpha \leq 7\ell + 6$;

(iii) $h = 3$ if $1 \leq \ell \leq 2$ and $\alpha = 7\ell + 4$.

Then $H^1(Z_4, \mathcal{L}_{Z_4}(-A_1 - A_2)) = 0$ and, in the same hypotheses as in part (ii) of Lemma 9.4, the restriction map $H^0(W_2, \mathcal{L}_{W_2}) \rightarrow H^0(\Delta, \mathcal{L}_\Delta)$ is surjective.

Proof. Suppose for a moment we proved that $H^1(Z_4, \mathcal{L}_{Z_4}(-A_1 - A_2)) = 0$. Look at the commutative diagram

$$\begin{array}{ccc} & & \mathcal{L}_{G_1+G_2}(-A_1 - A_2) \\ & & \downarrow \\ \mathcal{L}_{W_2} & \longrightarrow & \mathcal{L}_\Delta \\ \downarrow & & \downarrow \\ \mathcal{L}_{Z_4} & \longrightarrow & \mathcal{L}_{A_1+A_2} \end{array}$$

Notice that $H^0(G_1 + G_2, \mathcal{L}_{G_1+G_2}(-A_1 - A_2)) \simeq H^1(G_1 + G_2, \mathcal{L}_{G_1+G_2}(-A_1 - A_2)) = 0$ by degree reasons. Thus $H^0(\Delta, \mathcal{L}_\Delta) \simeq H^0(A_1 + A_2, \mathcal{L}_{A_1+A_2})$. Since the map $H^0(W_2, \mathcal{L}_{W_2}) \rightarrow H^0(Z_4, \mathcal{L}_{Z_4})$ is surjective by part (ii) of Lemma 9.4, we have that also $H^0(W_2, \mathcal{L}_{W_2}) \rightarrow H^0(\Delta, \mathcal{L}_\Delta)$ is surjective.

The proof that $H^1(Z_4, \mathcal{L}_{Z_4}(-A_1 - A_2)) = 0$ parallels the one of Lemma 9.2. The system $\mathcal{L}_{Z_4}(-A_1 - A_2)$ is Cremona equivalent to $\mathcal{L}(4\alpha - 3a; (\alpha - a)^6, [b - 2a + 1 - e, b - 2a]^2)$ and we are in the strong anticanonical case. Moreover the intersection of $\mathcal{L}_{Z_4}(-A_1 - A_2)$ with the canonical bundle is $2\ell - a + 2 < 0$.

First make a quadratic transformation based at the points of multiplicities $b - 2a + 1 - e, b - 2a + 1 - e, b - 2a$, getting the linear system

$$\mathcal{L}(8\alpha - 3b + 2e - 2; (\alpha - a)^6, b - 2a, (4\alpha + a - 2b - 1 + e)^2, 4\alpha + a - 2b - 2 + 2e).$$

Next make a quadratic transformation based at points of multiplicities $\alpha - a, \alpha - a, b - 2a$, getting the linear system

$$\begin{aligned} &\mathcal{L}(14\alpha - 7b + 4a + 4e - 4; (\alpha - a)^4, (7\alpha + 3a - 4b + 2e - 2)^2, \\ &6\alpha + 2a - 3b + 2e - 2, (4\alpha + a - 2b - 1 + e)^2, 4\alpha + a - 2b - 2 + 2e). \end{aligned}$$

Again, make a quadratic transformation based at points of multiplicities $\alpha - a, \alpha - a, 6\alpha + 2a - 3b + 2e - 2$, getting the linear system

$$\begin{aligned} &\mathcal{L}(20\alpha - 11b + 8a + 6e - 6; (\alpha - a)^2, 12\alpha + 6a - 7b + 4e - 4, \\ &(7\alpha + 3a - 4b + 2e - 2)^4, (4\alpha + a - 2b - 1 + e)^2, 4\alpha + a - 2b - 2 + 2e). \end{aligned}$$

Then, make a quadratic transformation based at the points of multiplicities $\alpha - a, \alpha - a, 12\alpha + 6a - 7b + 4e - 4$, getting the linear system

$$\begin{aligned} &\mathcal{L}(26\alpha - 15b + 12a + 8e - 8; (7\alpha + 3a - 4b + 2e - 2)^6, \\ &18\alpha - 11b + 10a + 6e - 6, (4\alpha + a - 2b - 1 + e)^2, 4\alpha + a - 2b - 2 + 2e). \end{aligned}$$

Set

$$x = 18\alpha - 11b + 10a + 6e - 6, y = 4\alpha + a - 2b - 1 + e, z = 4\alpha + a - 2b - 2 + 2e.$$

Note that $y = z - e + 1$, that

$$26\alpha - 15b + 12a + 8e - 8 = x + y + z - e + 1 = x + 2y$$

and

$$7\alpha + 3a - 4b + 2e - 2 = z - \ell - e$$

so that the system may be written as

$$\mathcal{L}(x + 2y; x, y^2, z, (z - \ell - e)^6).$$

Moreover one has $z - \ell - e = h - 2 \geq -1$, $z = \ell + h + e - 2 \geq \ell + e - 1 \geq -1$, and $y = z - e + 1 = \ell + h - 1 \geq \ell \geq 0$. In addition one computes

$$x = \frac{\alpha - 7\ell + e}{2} - 6 + h;$$

therefore $x \geq -1$ if and only if $\alpha + e \geq 7\ell + 10 - 2h$, which is equivalent to $\alpha \geq 7\ell + 9 - 2h$ (if equality holds one has $e = 1$). One checks that the hypotheses relating ℓ, α and h ensure that $x \geq -1$.

Suppose we are in case (i) or (ii). Then after splitting single (-1) -curves with no contribution to H^1 , the residual system is $\mathcal{L}(x + 2y; x, y^2, z)$, which is non-empty and non-special.

In case (iii) the relevant system is $\mathcal{L}(2\ell + 3; (\ell + 2)^2, \ell + 1, 1^6)$, which is non-empty and non-special in our cases. \square

Note that the previous lemma applies in all cases but six, which are

$$\ell = 0, 1 \leq \alpha \leq 4, \quad \text{corresponding to } (d, m) = (19\alpha, 6\alpha)$$

$$\ell = 1, 9 \leq \alpha \leq 10, \quad \text{corresponding to } (d, m) = (3 + 19\alpha, 1 + 6\alpha).$$

Finally we attach the planes $U_{1,4}$, and $U_{2,4}$ and Y_i , $i = 1, \dots, 4$, in order to obtain first W_4 and then W_5 . In this case the non-speciality is clear since the bundles there have degree e .

Proposition 9.6. *If $174/55 \leq d/m \leq 19/6$ and $(d, m) \neq (174, 55), (193, 61), (348, 110)$, then the system $\mathcal{L}(d; m^{10})$ has the expected dimension.*

Proof. Suppose (d, m) correspond to a pair (α, ℓ) satisfying any of the hypotheses of Lemma 9.5. Then we can take $a = \alpha - 2\ell - h$ as indicated by the lemma. If $\ell = 0, 1$ we need $a \geq 4\ell + 1$ in order to meet all the hypotheses of the previous lemmas. This requires $\alpha \geq 6\ell + 1 + h$ which is true in all cases, and one concludes the proof using the results of §5. If $\ell = 2$ we need $a \geq 12$, and $\alpha - 4 - h \geq 12$ in all cases, except for $(d, m) = (348, 110)$, i.e. $\alpha = 18$. If $\ell > 2$ we need $a > 4\ell + 5$, and $\alpha - 2\ell - h > 4\ell + 5$ in all cases.

Finally the cases $(d, m) = (19\alpha, 6\alpha)$ with $\alpha \leq 4$ are covered by the results in [4], [2], [9] and [10]. \square

10 The remaining cases

In this section we prove non-speciality of the three bundles left out in Proposition 9.6 and this will finish the proof of Theorem 0.1.

10.1 The case $d=174$, $m=55$.

The non-speciality of the linear system $\mathcal{L}(174; 55^{10})$ does not follow directly from the above arguments, and must be proved with an ad hoc argument, which however uses the above setting. In particular we make an analysis with the third degeneration (see §7).

Note that the virtual dimension is -1 , and therefore we have to prove that the system is empty. We argue by contradiction and suppose this is not the case. Then, by fixing extra base points, we may assume that the dimension of the linear system is zero on the general fibre of our degeneration. Thus the curve on the general fibre describes a surface in the total space of the degeneration, excluding the central fiber. By taking the closure of this surface, one finds a surface S which intersects the central fibre along a curve. We then conclude that there is some line bundle \mathcal{L} on the total space of the degeneration, i.e. the line bundle determined by S , which is a limit line bundle of $\mathcal{L}(174; 55^{10})$ on the central fibre, and such that the general section of the restriction of \mathcal{L} to the central fibre does not vanish identically on any irreducible component of the central fibre. We will then say that the limit line bundle $\mathcal{L}|_{X_0}$ is *centrally effective*.

In particular, in the setting of §7, there must be an integer a for which the corresponding limit line bundle is centrally effective. We will prove emptiness of $\mathcal{L}(174; 55^{10})$ by showing that there is no such an a .

For $d = 174$ and $m = 55$, we have $c = 87$, $e = 0$, $\alpha = 9$, $\ell = r = s = 1$, and $b = a + 14$. In the third degeneration, this gives the following bundles:

- $\mathcal{L}_{\tilde{V}_3} = \mathcal{L}(9a - 2; (4a - 1)^4, [2a, 2a]^2)$;
- $\mathcal{L}_{Z_3} = \mathcal{L}(36 - 3a; (9 - a)^6, [14 - a, 14 - a]^2)$;
- $\mathcal{L}_{T_3} = \mathcal{L}(28 - 2a; [14 - a, 14 - a]^2)$;
- $\mathcal{L}_{U_{i,3}} = \mathcal{O}$ for $i = 1, 2$.

On V_3 , the four quartics

$$\mathcal{L}(4; 2, 2, 2, 1, [1, 1]^2), \quad \mathcal{L}(4; 2, 2, 1, 2, [1, 1]^2), \quad \mathcal{L}(4; 2, 1, 2, 2, [1, 1]^2), \quad \mathcal{L}(4; 1, 2, 2, 2, [1, 1]^2)$$

each have intersection number -1 with the system $\mathcal{L}_{\tilde{V}_3}$, and so split off once each; the residual system is $\mathcal{L}'_{\tilde{V}_3} = \mathcal{L}(9a - 18; (4a - 8)^4, [2a - 4, 2a - 4]^2)$.

The four quartics each meet the self-double curve twice (once each on each side) and no other double curves on V_3 .

Recalling the proof of Lemma 9.1, we see that the system $\mathcal{L}'_{\tilde{V}_3}$ is Cremona equivalent to the system of curves of degree $a - 2$. In order to give a divisor on \tilde{V}_3 , the curves in the system must match with the four quartics; this requires that they meet the self-double curve in the eight fixed points that the four quartics do, four on each side.

Note that the self-double curve, which on \tilde{V}_3 pulls back to $F_1 + F_2$, is Cremona equivalent to the curve $\Gamma = \Gamma_1 + \Gamma_2$ we met in §8.2. The eight points correspond to the eight points t_i, s_i , $i = 1, \dots, 4$, on Γ , four on each of the conics Γ_1 and Γ_2 . We denote by D the divisor formed by these eight points on Γ . Lemma 8.1 says that D is not a bicanonical divisor on Γ . There is no curve of degree $a - 2$ containing D unless $a \geq 5$ and, for $a \geq 5$, D gives eight independent conditions to $\mathcal{L}(a - 2)$. Hence we must have $a \geq 5$ for the limit bundle to be centrally effective, in particular effective on V_3 .

For $a = 5$ the hypothesis in the transversality Lemma 8.2 is still met. Indeed $\mathcal{L}(3)(-D)$ is a pencil of cubics. What one has to show is that it does not cut out on Γ_1 or Γ_2 , off the base points in D , a pencil of degree 2 with ramification points at p_1, p_2 or q_1, q_2 . Suppose this is the case for all choices of the projective transformation ω as in §8.2. Then, as in the proof of Lemma 8.1, for some special ω , one of the points s_1, \dots, s_4 coincides with one of the points a_1, \dots, a_4 . Then Λ has a base point there, which is different from p_1 and q_1 , a contradiction. Therefore, if $a = 5$, this transversality implies that there are no divisors in $\mathcal{L}(3)(-D)$ which may correspond to Cartier divisors on V_3 . In conclusion we must have $a \geq 6$ for central effectivity.

The system on Z_3 is Cremona equivalent to $\mathcal{L}(36 - 3a; (9 - a)^6, [14 - a, 14 - a]^2)$ and, after the Cremona transformations, the two compound points are not collinear. Thus if $28 - 2a > 36 - 3a$ (i.e., if $a > 8$) the two tangent lines to E at these two points split. If so, the system is empty, since the two lines become (-1) -curves which meet. Thus we must have $a \leq 8$ for the limit bundle to be centrally effective, in particular effective on Z_3 .

The system on T_3 corresponds to a pencil of conics, which, as we saw in §8.1, sets up a correspondence between the two curves A_1 and A_2 , the intersection of T_3 with Z_3 . The

divisors in \mathcal{L}_{Z_3} which correspond to divisors on the central fibre must cut A_1 and A_2 in corresponding points.

Return now to the linear system on \tilde{V}_3 equivalent to $\mathcal{L}(a-2)(-D)$, formed by all curves of degree $a-2$ containing D . For $a \geq 6$ this system cuts out on either one of the conics Γ_1 and Γ_2 a complete linear series. This means that on \tilde{V}_3 the linear system $\mathcal{L}_{\tilde{V}_3}$ minus the four quartics and passing through the points on F_1 and F_2 corresponding to $t_i, s_i, i = 1, \dots, 4$, cuts out on either one of the curves F_1 and F_2 , off these base points, a complete linear series. Then we may apply Lemma 8.2, and conclude that curves in $\mathcal{L}(a-2)(-D)$ which may correspond to Cartier divisors on V_3 form a system of dimension $(a-2)(a+1)/2 - 8 - (2a-8) = (a^2 - 5a - 2)/2$.

Suppose we can apply transversality as indicated in §8.3. Then there are $14-a$ additional conditions on the system on Z_3 in order to obtain divisors which match with curves on T_3 and hence have a chance to be Cartier on the central fibre. Thus the dimension δ of the family of these divisors is at most

$$\delta = \max\{-1, \dim(\mathcal{L}(36-3a; (9-a)^6, [14-a, 14-a]^2)) - 14 + a\} \quad (10.1)$$

and central effectivity requires $\delta \geq 0$.

We note that the curves of this system, as well as the ones on V_3 , restrict to the double curve $E = V_3 \cap Z_3$ (which on Z_3 in this form is a nodal cubic through the six $(9-a)$ -points) in degree $a-2$.

We take up the three cases for a in turn and verify the needed hypothesis for transversality.

If $a = 8$ the system on Z_3 is Cremona equivalent to $\mathcal{L}(12; 1^6, [6, 6]^2)$. By making the obvious Cremona transformations, it can be further reduced to $\mathcal{L}(0; (-1)^6, 0^4)$, which means that it consists of six disjoint (-1) -curves. In this process the curves B_1 and A_1 are respectively mapped to a line M through three of the last four points and to a cubic N double at one of these points and passing through the first six base points. Since these six points are general, none of them coincides with the intersection of N and M off the common base point. This suffices to apply Lemma 8.2 and we conclude that the system is empty because then $\delta = -1$.

If $a = 7$, the system on Z_3 is Cremona equivalent to $\mathcal{L}(3; 1^4)$, which has dimension 5. Again the generality assumptions provide the required hypothesis for the transversality and we conclude as above since $\delta = -1$.

Finally let $a = 6$. By Lemma 7.5, the linear system \mathcal{L}_{V_3} on V_3 is non-special of dimension 2. Since for $a = 5$ the linear system on V_3 is empty, then the kernel linear system $\mathcal{L}_{V_3}(-E)$ is empty, and therefore the linear system cut out by \mathcal{L}_{V_3} on E has dimension 2.

The line bundle on Z_3 is Cremona equivalent to $\mathcal{L}(6; 1^6, 2^4)$, which is non special, of dimension 9, (see also by Lemma 9.2). This cuts out a complete linear series on the cubic N since the kernel linear system has dimension zero. Hence again the transversality holds and we conclude that $\delta = 1$ by (10.1).

The kernel linear system to E is empty (see the analysis of the $a = 7$ case), hence the linear system cut out on E has dimension $r \leq 1$.

There is no matching possible between this r -dimensional linear system of degree 4 on E , and the dimension 2 linear system restricted from the V_3 side, by generality of the construction. The curve E is a nodal rational curve, and we may embed its normalization

into \mathbb{P}^4 as a rational normal curve of degree 4; we have the complete g_4^4 on it which is cut out by the hyperplanes in \mathbb{P}^4 . Let x_1 and x_2 be the two points on the curve that are identified to form the nodal curve E . Consider the chordal line passing through the two points. Each point on this line gives rise to a complete g_4^3 which identifies the two points. The line minus the two points may be identified with the natural \mathbb{C}^* component of the Picard group of E . The two sublinear systems that we have correspond to a line and a plane meeting the chord, giving rise to a g_4^1 and g_4^2 identifying the points. The usual generality and transversality arguments as in §8, imply that the above line and plane are general under the condition of meeting the chord. Hence there is no hyperplane containing these both, and so there is no common restriction.

This completes the analysis of the $\mathcal{L}(174; 55^{10})$ system, and we conclude that it is empty as expected.

10.2 The case $d=193$, $m=61$.

Here $\ell = 1$ again and the virtual dimension is 4. The analysis, as in the previous case, uses the third degeneration and the four quartics split off once each.

Using $a = 7$ we see that:

- the linear system on \tilde{V}_3 Cremona reduces to $\mathcal{L}(5; 1^8)$, where the base points lie, as above, on Γ , which has dimension 12;
- the linear system on Z_3 Cremona reduces to $\mathcal{L}(6; 2^3, 1^7)$, which has dimension 11;
- the linear system on T_3 is composed with 8 conics plus a fixed line.

All systems are non-empty and non-special, the transversality criteria all hold. Indeed, from part (ii) of Lemma 9.4, we see that the system of curves on V_4 cuts out a complete linear series on the double curve E . Moreover one directly verifies that \mathcal{L}_{Z_3} cuts out complete linear series on the curves $A_i + B_i$, for $i = 1, 2$. Therefore the dimension of the system on V_3 is 6. Similarly the dimension of the system on $Z_3 + T_3$ is 2 and moreover its restriction of the latter system to E is injective. The restriction of the V_3 system to E has dimension 3. Using the same analysis as in the previous case, we see that these two series on E intersect in a linear series of dimension 1, and this implies that the family of matching curves on the central fibre has dimension 4.

10.3 The case $d=348$, $m=110$.

This is the double of $\mathcal{L}(174; 55^{10})$, its virtual dimension is 24, and we can prove it is non-special using the same ideas as above. Actually the proof is easier and we will be brief.

We have $c = 174$, $e = 0$, $\alpha = 18$, $\ell = 2$, $r = 1$, $s = 0$, and $b = a + 28$.

Consider the fourth degeneration: this is necessary since the quartics split with multiplicity 2 from the bundle on V_3 in the third degeneration. Then we have the following bundles (see §9):

- $\mathcal{L}_{\tilde{V}_4}$, which is Cremona equivalent to $\mathcal{L}(a - 4; [1, 1]^8)$ and the eight compound simple points lie four each on the two conics Γ_1, Γ_2 ;

- \mathcal{L}_{Z_4} which is Cremona equivalent to $\mathcal{L}(72 - 3a; (18 - a)^6, [28 - a, 28 - a]^2)$. This is in turn Cremona equivalent to a system of the form $\mathcal{L}(48 - 3a; (16 - a)^4, (14 - a)^6)$, where the six points of multiplicity $14 - a$ are general, whereas the four points of multiplicity $16 - a$ are not; three of them are on the same line L and two of these are infinitely near giving a compound $[16 - a, 16 - a]$ point; the line L is the image of one of the curves G_i , the other curve G_{3-i} is contracted to the compound $[16 - a, 16 - a]$ point;
- \mathcal{L}_{T_4} is Cremona equivalent to $\mathcal{L}(56 - 2a; [28 - a, 28 - a]^2)$;
- $\mathcal{L}_{U_{i,4}}$ and \mathcal{L}_{Y_j} are trivial, for $i = 1, 2$ and $j = 1, \dots, 4$.

We fix $a = 14$. Then $\dim(\mathcal{L}(10; [1, 1]^8)) = 49$. In order to compute the dimension of \mathcal{L}_{V_4} , we have to take into account the matching conditions with the self double curves. The usual transversality arguments tell us that this imposes at least 16 conditions, i.e., 12 for the matching on F_1 and F_2 and 4 for the matching on the curves $H_i, H'_i, i = 1, \dots, 4$ (see §9). Hence we find $\dim(\mathcal{L}_{V_4}) \leq 33$. In addition, from part (ii) of Lemma 9.4, we see that the system of matching curves on V_4 cuts out a complete linear series on the double curve E .

The system \mathcal{L}_{Z_4} is Cremona equivalent to $\mathcal{L}(6; 2^4)$. The base points are not general, but the above description implies that the system is non-special, of dimension 15. The usual transversality gives us 14 more matching conditions along the curves A_1, A_2 , on each of which \mathcal{L}_{Z_4} cuts out a complete linear series. Hence the system on $Z_4 + T_4$ has dimension 1.

We claim that the dimension of the system of matching curves on $V_4 \cup Z_4 \cup T_4$ is 24. Indeed such a curve can be constructed as follows: take any curve D_1 on \mathcal{L}_{Z_4} , which depends on one parameter, and add any curve D_2 on Z_4 such that D_1 and D_2 cut out the same divisor on E . Since E is a rational curve of arithmetic genus 1, $\mathcal{L}_{Z_4} \cdot E = \mathcal{L}_{V_4} \cdot E = 10$, and the matching curves on V_4 cut out a complete linear series on E , we see that D_2 varies with at most 23 parameters. This proves our claim.

Finally, adding the remaining surfaces $U_{i,4}, Y_j$ for $i = 1, 2$ and $j = 1, \dots, 4$ does not increase the number of parameters of matching curves. This gives an upper bound of 24 for the dimension of the system on the central fibre, thus proving non-speciality in this case.

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