

A RIEMANN-ROCH THEOREM FOR EDGE-WEIGHTED GRAPHS

RODNEY JAMES AND RICK MIRANDA

CONTENTS

1.	Introduction	1
2.	Change of Rings	3
3.	Reduction to $\mathbb{Q}$ -graphs	5
4.	Scaling	6
5.	Reduction to $\mathbb{Z}$ -graphs	8
	References	9

1. INTRODUCTION

The purpose of this article is to prove a Riemann-Roch theorem for edge-weighted graphs, inspired by (and extending) the theorem of Baker and Norine (see [1]). In that context, graphs without loops but with multiple edges are considered. We consider the existence of multiple edges to be equivalent to assigning to each pair of vertices an integral weight which records the number of edges between them. In our setting we consider arbitrary positive real numbers as edge weights. This variation forces several interesting adjustments to be made to the theory.

Let R be a subring of the real numbers $\mathbb{R}$. An R -graph G is a finite connected graph (without loops or multiple edges) where each edge is assigned a weight, which is a positive element of R . If we let the n vertices of G be $\{v_1, \dots, v_n\}$, we will denote by $p_{ij} = p_{ji}$ the weight of the edge joining v_i and v_j . If there is no edge connecting v_i and v_j , we set $p_{ij} = p_{ji} = 0$.

We define the *degree* of a vertex v_j of G to be the sum of the weights of the edges incident to it:

$$\deg(v_j) = \sum_{i \neq j} p_{ij}.$$

The *edge matrix* P of G is the symmetric $n \times n$ matrix defined by

$$(P)_{ij} = \begin{cases} -p_{ij} & \text{if } i \neq j \\ \deg(v_j) & \text{if } i = j. \end{cases}$$

The *genus* of G is defined as

$$g = \sum_{i < j} p_{ij} - n + 1.$$

An R -divisor D on G is a formal sum

$$D = \sum_{i=1}^n d_i \cdot v_i$$

1

where each $d_i \in R$; the divisors form a free R -module $\text{Div}(G)$ of rank n . We write $D_1 \geq D_2$ if the inequality holds at each vertex; for a constant c , we write $D \geq c$ (respectively $D > c$) if $d_i \geq c$ (respectively $d_i > c$) for each i .

The *degree* of a divisor D is

$$\deg(D) = \sum_{i=1}^n d_i$$

and the *ceiling* of D is the divisor

$$\lceil D \rceil = \sum_{i=1}^n \lceil d_i \rceil \cdot v_i.$$

The degree map is a homomorphism from $\text{Div}(G)$ to R , and the kernel $\text{Div}_0(G)$ of divisors of degree zero is a free R -module of rank $n - 1$.

Let $H_j = \deg(v_j) \cdot v_j - \sum_{i \neq j} p_{ij} \cdot v_i$, and set $\text{PDiv}(G) = \{\sum_{i=1}^n c_i H_i \mid c_i \in \mathbb{Z}\}$ to be the free $\mathbb{Z}$ -module generated by the H_j . (Note that the H_j divisors correspond to the columns of the matrix P .) If G is connected, $\text{PDiv}(G)$ has rank $n - 1$. Note that $\text{PDiv}(G) \subset \text{Div}_0(G)$; the quotient group is called the *Jacobian* of G .

For two divisors $D, D' \in \text{Div}(G)$, we say that D is *linearly equivalent* to D' , and write $D \sim D'$, if and only if $D - D' \in \text{PDiv}(G)$.

The *linear system* associated with a divisor D is

$$|D| = \{D' \in \text{Div}(G) \mid D \sim D' \text{ with } \lceil D' \rceil \geq 0\} = \{D' \in \text{Div}(G) \mid D \sim D' \text{ with } D' > -1\}.$$

We note that linearly equivalent divisors have the same linear system. The use of the ceiling divisor in the definition above is the critical difference between this theory and the integral theory developed by Baker and Norine [1].

The essence of the Riemann-Roch theorem, for divisors on algebraic curves, is to notice that the linear system corresponds to a vector space of rational functions, and to relate the dimensions of two such vector spaces. In our context we do not have vector spaces; so we measure the size of the linear system in a different way (as does Baker and Norine).

Define the h^0 of an R -divisor D to be

$$h^0(D) = \min\{\deg(E) \mid E \text{ is an } R\text{-divisor, } E \geq 0 \text{ and } |D - E| = \emptyset\}.$$

Note that $h^0(D) = 0$ if and only if $|D| = \emptyset$, and that linearly equivalent divisors have the same h^0 .

The *canonical divisor* of G is defined as

$$K = \sum (\deg(v_i) - 2) \cdot v_i.$$

The Riemann-Roch result that we will prove can now be stated.

Theorem 1.1. *Let G be a connected R -graph as above, and let D be an R -divisor on G . Then*

$$h^0(D) - h^0(K - D) = \deg(D) + 1 - g.$$

The results of Baker and Norine (see [1]) are exactly that the above theorem holds in the case of the subring $R = \mathbb{Z}$. Our proof depends on the Baker-Norine Theorem in a critical way; it would be interesting to provide an independent proof.

In [2] and [3], a Riemann-Roch theorem is proved for metric graphs with integral divisors; these results differ from the present result in two fundamental ways. First, our edge weights p_{ij} and the coefficients of the divisors are elements of the ring R . Second, the genus g is in R for the present result, whereas in [2] and [3], g is a nonnegative integer.

As an example, consider the R -graph G with two vertices and edge weight $p > 0$. For convenience, we will write the divisor $a \cdot v_1 + b \cdot v_2$ as the ordered pair (a, b) . The principal divisors are $\text{PDiv}(G) = \{(np, -np) \mid n \in \mathbb{Z}\}$, and $K = (p-2, p-2)$, with $g = p-1$. Note that if $p < 1$, we have $g < 0$.

For $(a, b) \in \text{Div}(G)$, the linear system $|a, b|$ can be written as

$$\begin{aligned} |(a, b)| &= \{(c, d) \in \text{Div}(G) \mid \lceil(c, d) \rceil \geq 0 \text{ and } (c, d) \sim (a, b)\} \\ &= \{(a + np, b - np) \mid n \in \mathbb{Z}, a + np > -1, b - np > -1\}. \end{aligned}$$

In what follows, we will be brief, and leave most of the details to the reader to verify. Once can check that $|a, b| \neq \emptyset$ if and only if $\lceil(1+a)/p\rceil + \lceil(1+b)/p\rceil \geq 2$.

The value of $h^0((a, b))$ can be computed as follows:

$$h^0((a, b)) = \begin{cases} 0 & \text{if } \lfloor(a+1)/p\rfloor + \lfloor(b+1)/p\rfloor < 0 \\ \min\{a+1-p\lfloor(a+1)/p\rfloor, b+1-p\lfloor(b+1)/p\rfloor\} & \text{if } \lfloor(a+1)/p\rfloor + \lfloor(b+1)/p\rfloor = 0 \\ a+b-p+2 & \text{if } \lfloor(a+1)/p\rfloor + \lfloor(b+1)/p\rfloor > 0 \end{cases}$$

Note that if $D = (a, b) \in \text{Div}(G)$ then $K - D = (p-2-a, p-2-b)$. To check that the Riemann-Roch theorem holds for D , it is easiest to consider the three cases (noted above) for the formula for $h^0((a, b))$. We note that (a, b) is in one of the three cases if and only if $(p-2-a, p-2-b)$ is in the opposite case. It is very straightforward then to check Riemann-Roch in case $\lfloor(a+1)/p\rfloor + \lfloor(b+1)/p\rfloor \neq 0$; one of the two h^0 values is zero. It is a slightly more interesting exercise, but still straightforward, to check it in case $\lfloor(a+1)/p\rfloor + \lfloor(b+1)/p\rfloor = 0$.

Unfortunately, this method of direct computation becomes intractable for R -graphs with $n > 2$.

2. CHANGE OF RINGS

Note that in the definition of the h^0 of a divisor, the minimum is taken over all non-negative R -divisors. Therefore, a priori, the definition of h^0 depends on the subring R . We note that if $R \subset S \subset \mathbb{R}$ are two subrings of $\mathbb{R}$, then any R -graph G and R -divisor D on G is also an S -graph and an S -divisor. In this section we will see that the h^0 in fact does not depend on the subring.

Any $H \in \text{PDiv}(G)$ can be written as an integer linear combination of any $n-1$ elements of the set $\{H_1, H_2, \dots, H_n\}$. If we exclude H_k , for example, then there are $n-1$ integers $\{m_j\}_{j \neq k}$ such that $H = \sum_{j \neq k} m_j H_j$, and we can write $H = \sum_{i=1}^n h_i \cdot v_i$ where

$$(2.1) \quad h_i = \begin{cases} m_i \deg(v_i) - \sum_{j \neq k, i} m_j p_{ij} & \text{if } i \neq k \\ -\sum_{j \neq k} m_j p_{jk} & \text{if } i = k. \end{cases}$$

Let P_k be the $(n-1) \times (n-1)$ matrix obtained by deleting the k th row and column from the matrix P . We can write the h_i 's other than h_k in matrix form as $\mathbf{h} = P_k \mathbf{m}$ where $\mathbf{h} = (h_i)_{i \neq k}$ and $\mathbf{m} = (m_i)_{i \neq k}$ are the corresponding column vectors.

For any $\mathbf{x} = (x_i) \in \mathbb{R}^{n-1}$ and $c \in \mathbb{R}$, we say $\mathbf{x} \geq c$ if and only if $x_i \geq c$ for each i ; similarly for a matrix $A = (a_{ij})$, We write $A \geq c$ if and only if $a_{ij} \geq c$ for each i, j .

Lemma 2.2. *If $\mathbf{x} = (x_i)_{i \neq k}$ is a column vector in $\mathbb{R}^{n-1}$ such that $P_k \mathbf{x} \geq 0$, then $\mathbf{x} \geq 0$. Furthermore, P_k is nonsingular and $P_k^{-1} \geq 0$.*

Proof. Let $V_i = \{i' \mid p_{ii'} > 0, i' \neq k, i' \neq i\}$ be the set of indices of vertices connected to v_i (excluding k). Suppose that it is the case that $x_i < 0$, and that $x_i \leq x_{i'}$ for all $i' \in V_i$. Then

$$\begin{aligned} (P_k \mathbf{x})_i &= x_i \deg(v_i) - \sum_{i' \in V_i} x_{i'} p_{ii'} \\ &= x_i p_{ik} + x_i \sum_{i' \in V_i} p_{ii'} - \sum_{i' \in V_i} x_{i'} p_{ii'} \\ &= x_i p_{ik} + \sum_{i' \in V_i} p_{ii'} (x_i - x_{i'}), \end{aligned}$$

and we note that with our assumptions, no term here is positive. Since the sum is non-negative, we conclude that all terms are zero. We have verified the following therefore, if $P_k \mathbf{x} \geq 0$:

$$(2.3) \quad x_i < 0 \text{ and } x_i \leq x_{i'} \text{ for all } i' \in V_i \Rightarrow p_{ik} = 0 \text{ and } x_i = x_{i'} \text{ for all } i' \in V_i.$$

Now assume that $\mathbf{x} \not\geq 0$; then there is an index j such that $x = x_j < 0$ and $x_j \leq x_i$ for all $i \neq k$. By (2.3), we conclude that $x_i = x$ for all $i \in V_j$, and also that $p_{jk} = 0$. We see, by induction on the distance in G to the vertex v_j , that we must have $x_i = x$ and $p_{ij} = 0$ for all $i \neq k$. This contradicts the connectedness of G : vertex v_k has no edges on it. This proves the first statement.

Now suppose that $\mathbf{x} \in \ker P_k$; then $\mathbf{x} \geq 0$. Also, $-\mathbf{x} \in \ker P_k$, and thus $-\mathbf{x} \geq 0$; we conclude that $\mathbf{x} = \mathbf{0}$. Hence $\ker P_k = \{\mathbf{0}\}$ and P_k is invertible.

Let $\mathbf{y} = P_k \mathbf{x}$. Since $\mathbf{y} \geq 0 \Rightarrow \mathbf{x} \geq 0$ and P_k is invertible, $\mathbf{x} = P_k^{-1} \mathbf{y} \geq 0$ for all $\mathbf{y} \geq 0$. Applying $\mathbf{y} = \mathbf{e}_i$ for each $i \neq k$, where $(\mathbf{e}_i)_j = 1$ for $i = j$ and 0 otherwise, we have $P_k^{-1} \geq 0$. $\square$

We can now prove the main result for this section.

Proposition 2.4. *Suppose that all of the entries of the matrix P are in two subrings R and R' , and that all the coordinates of the divisor D are also in both R and R' . Then (using the obvious notation) $h^0 = h^{0'}$.*

Proof. It suffices to prove the statement when one of the subrings is R and the other is $\mathbb{R}$. In this case we'll use the notation Rh^0 and $\mathbb{R}h^0$, respectively, for the two minima in question.

First note that the linear system $|D|$ is clearly independent of the ring; and in particular, whether a linear system is empty or not is also independent.

Therefore, the minimum in question for the $\mathbb{R}h^0$ computation is over a strictly larger set of divisors; and hence there can only be a smaller minimum. This proves that $Rh^0(D) \geq \mathbb{R}h^0(D)$.

Suppose that E is an $\mathbb{R}$ -divisor, $E \geq 0$, and $|D - E| = \emptyset$, achieving the minimum, so that $\mathbb{R}h^0(D) = \deg(E)$. If E is an R -divisor, it also achieves the minimum in R and $Rh^0(D) = \mathbb{R}h^0(D)$. We will show that in fact E must be an R -divisor.

Now suppose that E is not an R -divisor, and write $D = \sum_{i=1}^n d_i \cdot v_i$ and $E = \sum_{i=1}^n e_i \cdot v_i$, with k the index of an element such that $e_k \notin R$. Since $\mathbb{R}h^0(D) = \deg(E)$, for any $\epsilon \in \mathbb{R}$ with $0 < \epsilon \leq e_k$, we have that $E - \epsilon \cdot v_k \geq 0$, and therefore $|D - E + \epsilon \cdot v_k| \neq \emptyset$. Hence there are principal divisors H such that $D - E + \epsilon \cdot v_k + H > -1$.

Let $\mathcal{H}_\epsilon$ be the set of all such H ; by assumption, this is a nonempty set. Note that if $H \in \mathcal{H}_\epsilon$, and $H = \sum_{i=1}^n h_i \cdot v_i$, then $d_i - e_i + h_i > -1$ for each $i \neq k$, and

$$(2.5) \quad d_k - e_k + \epsilon + h_k > -1.$$

Also, since $|D - E| = \emptyset$, there is a k' such that $d_{k'} - e_{k'} + h_{k'} \leq -1$; combined with the conditions above, the only possibility is $k' = k$. Since $d_k \in R$, $h_k \in R$ and $e_k \notin R$, $d_k - e_k + h_k \neq -1$, and thus $d_k - e_k + h_k < -1$. Hence $-1 - \epsilon < d_k - e_k + h_k < -1$.

For any $H \in \mathcal{H}_\epsilon$, there are unique integers m_i such that $H = \sum_{i \neq k} m_i H_i$. Let $\mathbf{d} = (d_i)_{i \neq k}$, $\mathbf{e} = (e_i)_{i \neq k}$, and $\mathbf{m} = (m_i)_{i \neq k}$ be the corresponding column vectors, and define $\mathbf{f} = (f_i)_{i \neq k} = \mathbf{d} - \mathbf{e} + P_k \mathbf{m}$. Note that $\mathbf{f} > -1$, and $h_k = -\sum_{i \neq k} m_i p_{ik}$ by (2.1).

We can write $\mathbf{m} = P_k^{-1}(\mathbf{f} - \mathbf{d} + \mathbf{e})$, and by Lemma 2.2, $P_k^{-1} \geq 0$. Therefore, since $\mathbf{e} \geq 0$ and $\mathbf{f} > -1$, the m_i are bounded from below; set $M \leq m_i$ for all $i \neq k$.

We claim that, for $H = \sum_{i \neq k} m_i H_i \in \mathcal{H}_\epsilon$, the possible coordinates $h_k = -\sum_{i \neq k} m_i p_{ik}$ form a discrete set. It will suffice to show that, for any real x , the possible coordinates h_k which are at least $-x$ is a finite set.

To that end, for any $x \in \mathbb{R}$ set $\mathcal{H}_\epsilon(x) = \{H \in \mathcal{H}_\epsilon \mid \sum_{i \neq k} m_i p_{ik} \leq x\}$; for large enough x this set is nonempty.

Fix $x \in \mathbb{R}$ such that $\mathcal{H}_\epsilon(x) \neq \emptyset$ and choose $j \neq k$ such that $p_{jk} > 0$. For $H = \sum_{i \neq k} m_i H_i \in \mathcal{H}_\epsilon(x)$ we then have

$$M \leq m_j \leq \frac{x - \sum_{i \neq j, k} m_i p_{ik}}{p_{jk}} \leq \frac{x - M \sum_{i \in V_k, i \neq j} p_{ik}}{p_{jk}}.$$

Thus the coefficients $m_j \in \mathbb{Z}$ are bounded both below and above, and hence can take on only finitely many values. It follows that the set of possible values of $h_k = -\sum_{i \neq k} m_i p_{ik}$ is also finite, for $H \in \mathcal{H}_\epsilon(x)$. As noted above, this implies that these coordinates h_k , for $H \in \mathcal{H}_\epsilon$, form a discrete set. This in turn implies that there is a maximum value h for the possible h_k , since for all such we have $d_k - e_k + h_k < -1$.

Note that if $\epsilon < \epsilon'$, then $\mathcal{H}_\epsilon \subset \mathcal{H}_{\epsilon'}$.

We may now shrink ϵ (if necessary) to achieve $\epsilon < e_k - d_k - h - 1$. This gives a contradiction, since now $d_k - e_k + \epsilon + h_k \leq d_k - e_k + \epsilon + h < -1$ for $H \in \mathcal{H}_\epsilon$, violating (2.5). We conclude that E is in fact an R -divisor as desired, finishing the proof. $\square$

The result above allows us to simply consider the case of $\mathbb{R}$ -graphs.

At the other end of the spectrum, the case of $\mathbb{Z}$ -graphs is equivalent to the Baker-Norine theory.

The Baker-Norine dimension of a linear system associated with a divisor D on a graph G defined in [1] is equal to

$$r(D) = \min\{\deg(E) \mid E \in \text{Div}(G), E \geq 0 \text{ and } |D - E|_{BN} = \emptyset\} - 1$$

where here the linear system associated with a divisor D is

$$|D|_{BN} = \{D' \in \text{Div}(G) \mid D' \geq 0 \text{ and } D \sim D'\}.$$

If we are restricted to $\mathbb{Z}$ -divisors on $\mathbb{Z}$ -graphs, the h^0 dimension is compatible with the Baker-Norine dimension:

Lemma 2.6. *If G is a $\mathbb{Z}$ -graph and D a $\mathbb{Z}$ -divisor on G , then $h^0(D) = r(D) + 1$.*

Proof. Note that $\lceil D \rceil = D$ since each component of D is in $\mathbb{Z}$. This implies that $|D| = |D|_{BN}$ which gives the result. $\square$

3. REDUCTION TO $\mathbb{Q}$ -GRAPHS

Note that the definition of $h^0(D)$ depends on the coordinates of D and on the entries p_{ij} of the matrix P which give the edge-weights in the graph G . Indeed, the set $\mathcal{E}$ of divisors with empty linear systems depends continuously on P , as a subset of $\mathbb{R}^n$. (If $\mathcal{F}_0$ is the set of divisors D with $d_i > -1$ for each i , then $\mathcal{E}$ is the complement of the union of all the translates of $\mathcal{F}_0$ by the columns of P .)

The value of $h^0(D)$ is essentially the taxicab distance from D to $\mathcal{E}$. This also depends continuously on the coordinates of D .

Since $\mathbb{Q}$ is dense in $\mathbb{R}$, by approximating both P and D by rationals, we see that it will suffice to prove the Riemann-Roch theorem for $\mathbb{Q}$ -graphs:

Proposition 3.1. *Suppose that the Riemann-Roch Theorem 1.1 is true for connected $\mathbb{Q}$ -graphs. Then the Riemann-Roch Theorem is true for connected $\mathbb{R}$ -graphs.*

4. SCALING

Suppose that G is an R -graph, with edge weights p_{ij} . For any $a > 0$, $a \in R \subset \mathbb{R}$, define aG to be the R -graph with the same vertices, and edge weights $\{ap_{ij}\}$. In other words, if P defines G , then aG is the R -graph defined by the matrix aP .

We will use subscripts to denote which R -graph we are using to compute with, e.g., $|D|_G$, $h_G^0(D)$, etc. if necessary.

For any divisor D on G and $a > 0$, define

$$T_a(D) = aD + (a - 1)I$$

where

$$I = \sum_i 1 \cdot v_i.$$

The transformation T_a is a homothety by a , centered at $-I$.

Lemma 4.1. *Let D be an R -divisor. If $a, b > 0$ with $a, b \in R$, then the following hold:*

- (1) $T_b \circ T_a = T_{ab}$
- (2) $T_a(D + H) = T_a(D) + aH$
- (3) $[D] \geq 0 \Leftrightarrow [T_a(D)] \geq 0$
- (4) $|D|_G \neq \emptyset \Leftrightarrow |T_a(D)|_{aG} \neq \emptyset$
- (5) $|D - E|_G \neq \emptyset \Leftrightarrow |T_a(D) - aE|_{aG} \neq \emptyset$

Proof. (1) Suppose that $D = \sum_i d_i \cdot v_i$. Then:

$$\begin{aligned} T_a(T_b(D)) &= T_a\left(\sum_i (bd_i + b - 1) \cdot v_i\right) \\ &= \sum_i (a(bd_i + b - 1) + a - 1) \cdot v_i \\ &= \sum_i (abd_i + ab - a + a - 1) \cdot v_i \\ &= \sum_i (abd_i + ab - 1) \cdot v_i \\ &= T_{ab}(D). \end{aligned}$$

(2) Let $a > 0$ and $D, H \in \text{Div}(G)$, then

$$\begin{aligned} T_a(D + H) &= a(D + H) + (a - 1)I \\ &= aD + aH + (a - 1)I \\ &= T_a(D) + aH. \end{aligned}$$

(3) Let $D = \sum_i d_i \cdot v_i \in \text{Div}(G)$ and $a > 0$. Since $T_a(D) = \sum_i (ad_i + a - 1) \cdot v_i$, we have

$$\begin{aligned} \lceil T_a(D) \rceil \geq 0 &\Leftrightarrow ad_i + a - 1 > -1 \text{ for each } i \\ &\Leftrightarrow d_i > -1 \text{ for each } i \\ &\Leftrightarrow \lceil D \rceil \geq 0. \end{aligned}$$

(4) Suppose $|D|_G \neq \emptyset$. Then there is a $H \in \text{PDiv}(G)$ such that $\lceil D+H \rceil \geq 0$. Since $T_a(D+H) = T_a(D) + aH$ and $aH \in \text{PDiv}(aG)$, by part (3) we have $\lceil T_a(D) + aH \rceil \geq 0$ and thus $|T_a(D)|_{aG} \neq \emptyset$.

The converse is an identical argument.

(5) Let $D' = D - E$; then from (4), $|D'|_G \neq \emptyset \Leftrightarrow |T_a(D')|_{aG} \neq \emptyset$ where $T_a(D') = T_a(D - E) = T_a(D) - aE$.

□

Corollary 4.2. $h_{aG}^0(T_a(D)) = ah_G^0(D)$

Proof. Since $a > 0$, from Lemma 4.1 (5) we have

$$\begin{aligned} h_{aG}^0(T_a(D)) &= \min_{E' \in \text{Div}(aG)} \{\deg(E') \mid E' \geq 0, |T_a(D) - E'|_{aG} = \emptyset\} \\ &= \min_{E \in \text{Div}(G)} \{\deg(aE) \mid aE \geq 0, |T_a(D) - aE|_{aG} = \emptyset\} \\ &= a \left(\min_{E \in \text{Div}(G)} \{\deg(E) \mid E \geq 0, |T_a(D) - aE|_{aG} = \emptyset\} \right) \\ &= a \left(\min_{E \in \text{Div}(G)} \{\deg(E) \mid E \geq 0, |D - E|_G = \emptyset\} \right) \\ &= ah_G^0(D). \end{aligned}$$

□

Lemma 4.3. Let D be an R -divisor. If $a > 0$ with $a \in R$ then the following hold:

- (1) $K_{aG} = T_a(K_G) + (a - 1)I$
- (2) $K_{aG} - T_a(D) = T_a(K_G - D)$
- (3) $\deg(T_a(D)) = a \deg(D) + (a - 1)(n)$
- (4) $g_{aG} = ag_G + (a - 1)(n - 1)$.

Proof. (1) Since $K_{aG} = \sum_i (a \deg(v_i) - 2) \cdot v_i$, we have

$$\begin{aligned} T_a(K_G) &= T_a\left(\sum_i (\deg(v_i) - 2) \cdot v_i\right) \\ &= a \sum_i (\deg(v_i) - 2) \cdot v_i + \sum_i (a - 1) \cdot v_i \\ &= \sum_i (a \deg(v_i) - 2a + a - 1) \cdot v_i \\ &= \sum_i (a \deg(v_i) - a - 1) \cdot v_i \\ &= K_{aG} - (a - 1)I. \end{aligned}$$

(2)

$$\begin{aligned}
K_{aG} - T_a(D) &= T_a(K_G) + (a-1)I - T_a(D) \\
&= aK_G + (a-1)I + (a-1)I - aD - (a-1)I \\
&= a(K_G - D) + (a-1)I \\
&= T_a(K_G - D).
\end{aligned}$$

(3)

$$\begin{aligned}
\deg(T_a(D)) &= \deg(aD + (a-1)I) \\
&= a \deg(D) + (a-1) \deg(I) \\
&= a \deg(D) + (a-1)(n).
\end{aligned}$$

(4)

$$\begin{aligned}
g_{aG} &= \sum_i a p_{ij} - n + 1 \\
&= a \sum_i p_{ij} - an + a + (a-1)n + 1 - a \\
&= ag_G + (a-1)(n-1).
\end{aligned}$$

□

5. REDUCTION TO $\mathbb{Z}$ -GRAPHS

Theorem 5.1. *Let $a > 0$; then*

$$(5.2) \quad h_G^0(D) - h_G^0(K_G - D) = \deg(D) - g_G + 1$$

if and only if

$$(5.3) \quad h_{aG}^0(T_a(D)) - h_{aG}^0(K_{aG} - T_a(D)) = \deg(T_a(D)) - g_{aG} + 1.$$

Proof. Let $a > 0$. Multiplying (5.2) by a , we have

$$ah_G^0(D) - ah_G^0(K_G - D) = a \deg(D) - ag_G + a.$$

The left side of this equation is equal to

$$h_{aG}^0(T_a(D)) - h_{aG}^0(T_a(K_G - D)) = h_{aG}^0(T_a(D)) - h_{aG}^0(K_{aG} - T_a(D))$$

using Corollary 4.2 and Lemma 4.3 (2). The right side of the equation is

$$\deg(T_a(D)) - (a-1)(n) - g_{aG} + (a-1)(n-1) + a = \deg(T_a(D)) - g_{aG} + 1$$

using Lemma 4.3 (3) and (4). This proves that (5.2) implies (5.3); the converse is identical. □

Corollary 5.4. *Suppose that the Riemann-Roch Theorem 1.1 is true for connected $\mathbb{Z}$ -graphs. Then the Riemann-Roch Theorem is true for connected $\mathbb{Q}$ -graphs.*

Proof. Given a connected $\mathbb{Q}$ -graph G and a $\mathbb{Q}$ -divisor D on it, there is an integer $a > 0$ such that aG is a connected $\mathbb{Z}$ -graph and $T_a(D)$ is a $\mathbb{Z}$ -divisor. Therefore by hypothesis, the Riemann-Roch statement (5.3) holds. Hence by Theorem 5.1, (5.2) holds, which is the Riemann-Roch theorem for D on G . □

We now have the ingredients to prove Theorem 1.1.

Proof. First, we note again that the Riemann-Roch Theorem of [1] is equivalent to the Riemann-Roch theorem for connected $\mathbb{Z}$ -graphs in our terminology. Therefore, using Corollary 5.4, we conclude that the Riemann-Roch Theorem is true for connected $\mathbb{Q}$ -graphs. Then, using Proposition 3.1, we conclude that Riemann-Roch holds for connected $\mathbb{R}$ -graphs.

Finally, Proposition 2.4 finishes the proof of the Riemann-Roch theorem for divisors on arbitrary R -graphs, for any subring $R \subset \mathbb{R}$. $\square$

REFERENCES

- [1] Baker, Matthew and Norine, Serguei, Riemann-Roch and Abel-Jacobi Theory on a Finite Graph, *Advances in Mathematics* 215 (2007) 766-788.
- [2] Gathmann, Andreas and Kerber, Michael, A Riemann-Roch Theorem in Tropical Geometry, *Mathematische Zeitschrift* 259 (2008) 217-230.
- [3] Mikhalkin, Grigory and Zharkov, Ilia, Tropical Curves, Their Jacobians, and Theta Functions, *preprint* arXiv:math/0612267v2 [math.AG], 20 Nov 2007.

DEPARTMENT OF MATHEMATICS, COLORADO STATE UNIVERSITY, FORT COLLINS, CO 80523 USA