ON THE GEOMETRIC GENUS OF REDUCIBLE SURFACES AND DEGENERATIONS OF SURFACES TO UNIONS OF PLANES

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1. Introduction

In this paper we study some properties of degenerations of surfaces whose general fibre is a smooth projective algebraic surface and whose central fibre is a reduced, connected surface $X \subset \mathbb{P}^r$, $r \geqslant 3$, which is assumed to be a union of planes. Here we present a first set of results on the subject; other aspects are still work in progress and will appear later (see [4]).

Our original motivation has been a series of papers by Guido Zappa appeared in the 1940–50's regarding degenerations of scrolls to unions of planes and the computation of bounds for the topological invariants of an arbitrary smooth projective surface which is assumed to degenerate to a union of planes (see [18, 19, 20, 21, 22, 23, 24] and [4]).

Zappa was in turn motivated by earlier papers by Francesco Severi concerning *Zeuthen's problem*, i.e. the existence of degenerations of smooth projective (space) curves to unions of lines with only nodes as singularities (now called *stick curves*).

Zeuthen's problem has been studied by several authors, also recently (see e.g. [14]); on the contrary, unions of planes have been studied only in terms of degenerations of a few types of smooth surfaces, e.g. K3 surfaces (see [5, 6, 7, 9]).

In this paper, we first study the geometry and the combinatorics of a union of planes X considered as a reduced, connected surface on its own (cf. §3 and 4); then, we focus on the case in which X is the central fibre of an embedded degeneration $\mathfrak{X} \to \Delta$, where Δ is the complex unit disk and where $\mathfrak{X} \subseteq \Delta \times \mathbb{P}^r$, $r \geqslant 3$, is a closed subscheme of relative dimension two. In this case, we deduce some properties of the general fibre $\mathfrak{X}_t$, $t \neq 0$, of the degeneration from the ones of its central fibre $\mathfrak{X}_0 = X$ (see §5).

It is well-known that, in dimension one, for any integer $g \geqslant 2$ any smooth projective curve of genus g with general moduli and sufficiently general degree can be degenerated to a suitable stick curve (see, e.g. [1] and [17]).

On the contrary, in dimension two, worse singularities than normal crossings are needed in order to degenerate as many surfaces as possible to unions of planes (cf. [4]).

Here we shall focus on the case of X a union of planes — or more generally a union of smooth projective surfaces — whose singularities are:

- in codimension one, double curves which are smooth and irreducible;
- multiple points, which are locally analytically isomorphic to the vertex of a cone over a stick curve with arithmetic genus either zero or one and which is projectively normal in the projective space it spans.

These multiple points will be called *Zappatic singularities*, whereas a surface like X will be called a *Zappatic surface*. If moreover $X \subset \mathbb{P}^r$, for some positive r, and if all its irreducible components are planes, then X is said to be a *planar Zappatic surface*.

Actually we will concentrate on the so called *good Zappatic surfaces*, i.e. Zappatic surfaces having only Zappatic singularities whose associated stick curve has one of the following dual graphs (cf. Examples 2.8 and 2.9, Definition 3.6, Figures 2 and 4):

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R_n: a chain of length n, with $n \geqslant 3$;
S_n: a fork with $n-1$ teeth, with $n \geqslant 4$;
E_n: a cycle of order n, with $n \geqslant 3$.

Let us call R_n-, S_n-, E_n-*point* the corresponding multiple point of the Zappatic surface X. These singularities play a major role in the whole subject (cf. [4]).

We remark that a Zappatic surface X is locally Gorenstein (i.e. its dualizing sheaf ω_X is invertible) if and only if its Zappatic singularities are only E_n-points, for any $n \geqslant 3$.

We associate to a good Zappatic surface X a graph G_X (see Definition 3.7) which encodes the configuration of the irreducible components of X as well as its Zappatic singularities.

We shall see (cf. Sections 3 and 4) how to combinatorially compute from the associated graph G_X some intrinsic and extrinsic invariants of X, e.g. the Euler-Poincaré characteristic $\chi(\mathcal{O}_X)$, the geometric genus $p_g(X)$ (cf. Remark 3.4), as well as — when $X \subset \mathbb{P}^r$, $r \geqslant 3$ — the degree $d = \deg(X)$, the sectional genus g, and so on.

When X is further assumed to be the central fibre of a degeneration $\mathcal{X} \to \Delta$ (resp., of an embedded degeneration $\mathcal{X} \to \Delta$, where $\mathcal{X} \subset \Delta \times \mathbb{P}^r$, $r \geqslant 3$) we will then compute intrinsic (resp., intrinsic and extrinsic) invariants of the general fibre $\mathcal{X}_t$, for $t \neq 0$.

We shall see how to directly compute some of the invariants of X by means of the associated graph G_X. Determining formulas for a few invariants (e.g. d and g) is quite easy, whereas for other invariants, like $\chi(\mathcal{O}_X)$, it requires some more work.

Actually the computation of the geometric genus is still an open question, in general. Indeed, we first prove the following (cf. Theorem 4.15):

Theorem. *Let $X = \bigcup_i X_i$ be a Zappatic surface with global normal crossings, i.e. with only E_3-points as Zappatic singularities. Denote by ω_X the dualizing sheaf of X and by G_X the associated graph of X. Consider the natural map:*

$$\Phi : \bigoplus_i H^1(X_i, \mathcal{O}_{X_i}) \to \bigoplus_{i,j} H^1(C_{ij}, \mathcal{O}_{C_{ij}}),$$

where $C_{ij} = X_i \cap X_j$ (cf. formula (4.8)*). Then, the following inequality holds:*

$$p_g(X) := h^0(X, \omega_X) \leqslant h_2(G_X, \mathbb{C}) + \sum_{i=1}^{v} p_g(X_i) + \dim(\operatorname{coker}(\Phi)). \tag{1.1}$$

Furthermore, a sufficient condition for the equality in (1.1) *to hold is either that*

(i) *each irreducible component X_i is a regular surface (i.e. $h^1(\mathcal{O}_{X_i}) = 0$), or that*
(ii) *for any irregular component X_j of X, the divisor $C_j := X_j \cap \overline{(X \setminus X_j)}$ is ample on X_j.*

The proof of the above theorem also shows the following:

Corollary. *Let X be a planar Zappatic surface with global normal crossings (i.e. only E_3-points) and G_X be its associated graph. Then, there exists an explicit isomorphism*

$$H^0(X, \omega_X) \cong H_2(G_X, \mathbb{C}), \tag{1.2}$$

where ω_X is the dualizing sheaf of X. Therefore

$$p_g(X) := h^0(X, \omega_X) = h_2(G_X, \mathbb{C}).$$

From the proof of Theorem 4.15, it will be clear that the isomorphism (1.2) essentially follows from evaluation of residues at the E_3-points, with a suitable use of signs.

We show that equality holds in (1.1) when X is smoothable, i.e. when X is the central fibre of a *semistable degeneration* $\mathcal{X} \to \Delta$: this follows by the computation of the geometric genus $p_g(\mathcal{X}_t)$ of the general fibre $\mathcal{X}_t$, for $t \neq 0$, via the Clemens-Schmid exact sequence (cf. Theorem 5.12).

We remark that our computation of the geometric genus is independent of the fact that X is the central fibre of a semistable degeneration. We deal with this particular case in §5, where we show that in a semistable degeneration, whose central fibre is a Zappatic surface with only E_3-points, the geometric genus of the fibres is constant (see Corollary 5.15).

It is still an open problem to find an example for which the strict inequality holds in (1.1).

Finally, we will see that the above results can be generalized to a smoothable good Zappatic surface, i.e. with R_n-, S_n- and E_n-points, for any $n \geqslant 3$ (see Theorem 5.20).

A natural question to ask is which Zappatic singularities are needed in order to degenerate as many surfaces as possible. Results and some examples contained in [4] suggest that, even if a given projective surface X needs E_n-, R_n-, or S_n-points with large n, there might be a birational model of X which needs just R_3- and E_n-points, with $n \leqslant 6$. For example, in [6] there are interesting examples of K3 surfaces degenerating to a Zappatic surface with at most R_3- and E_6-points, called *pillow* degenerations. However, we do not have enough evidence to state a reasonable conjecture in this direction.

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2. Reducible curves and associated graphs

Let C be a projective curve and let C_i, $i = 1, \ldots, n$, be its irreducible components. We will assume that:

- C is connected and reduced;
- C has at most nodes as singularities;
- the curves $C_i, i = 1, \ldots, n$, are smooth.

If two components C_i, C_j, $i < j$, intersect at m_{ij} points, we will denote by P_{ij}^h, $h = 1, \ldots, m_{ij}$, the corresponding nodes of C.

We can associate to this situation a simple (i.e. with no loops), connected graph G_C:

- whose vertices $v_1, \ldots, v_n$, correspond to the components $C_1, \ldots, C_n$;
- whose edges η_{ij}^h, $i < j$, $h = 1, \ldots, m_{ij}$, joining the vertices v_i and v_j, correspond to the nodes P_{ij}^h of C.

We will assume the graph to be *lexicographically oriented*, i.e. each edge is assumed to be oriented from the vertex with lower index to the one with higher index.

We will use the following notation:

- v : the number of vertices of G_C, i.e. $v = n$;
- e : the number of edges of G_C, i.e. the number of nodes of C;
- g_i : the genus of the curve C_i, which we consider as the *weight* of the vertex v_i;
- $\chi(G_C) = v - e$ is the Euler-Poincaré characteristic of G_C;
- $h_1(G_C) = 1 - \chi(G_C)$ is the first Betti number of G_C.

Remark that conversely, given any simple, connected, weighted (oriented) graph G, there is some curve C such that $G = G_C$.

One has the following basic result:

Theorem 2.1. *In the above situation*

$$\chi(\mathcal{O}_C) = \chi(G_C) - \sum_{i=1}^{v} g_i = v - e - \sum_{i=1}^{v} g_i. \tag{2.2}$$

Proof. Let $\nu : \tilde{C} \to C$ be the normalization morphism; this defines the exact sequence of sheaves on C:

$$0 \to \mathcal{O}_C \to \nu_*(\mathcal{O}_{\tilde{C}}) \to \underline{\tau} \to 0, \tag{2.3}$$

where $\underline{\tau}$ is a sky-scraper sheaf supported at $\mathrm{Sing}(C)$. Since the singularities of C are only nodes, one easily determines $H^0(C, \underline{\tau}) \cong \mathbb{C}^e$. Therefore, by the exact sequence (2.3), one gets

$$\chi(\mathcal{O}_C) = \chi(\nu_*(\mathcal{O}_{\tilde{C}})) - e.$$

By the Leray isomorphism and by the fact that ν is finite, one has $\chi(\nu_*(\mathcal{O}_{\tilde{C}})) = \chi(\mathcal{O}_{\tilde{C}})$. Since $\tilde{C}$ is a disjoint union of the $v = n$ irreducible components of C, one has $\chi(\mathcal{O}_{\tilde{C}}) = v - \sum_{i=1}^{v} g_i$, which proves (2.2). (Cf. also [2] for another proof.) □

We remark that formula (2.2) is equivalent to

$$p_a(C) = h_1(G_C) + \sum_{i=1}^{v} g_i, \tag{2.4}$$

(cf. Proposition 3.12).

Notice that C is locally Gorenstein, i.e. the dualizing sheaf ω_C is invertible. One defines the *geometric genus* of C to be

$$p_g(C) := h^0(C, \omega_C). \tag{2.5}$$

By the Riemann-Roch Theorem, one has

$$p_g(C) = p_a(C) = h_1(G_C) + \sum_{i=1}^{v} g_i = e - v + 1 + \sum_{i=1}^{v} g_i. \tag{2.6}$$

However, one can prove the previous formula combinatorially by showing that there is a natural short exact sequence:

$$0 \to \bigoplus_{i=1}^{v} H^0(C_i, \omega_{C_i}) \to H^0(C, \omega_C) \to H_1(G_C, \mathbb{C}) \to 0 \tag{2.7}$$

We will not dwell on this now, since we shall show the existence of an analogous sequence in the surface case in §4.

If we have a flat family $\mathcal{C} \to \Delta$ over a disc Δ with general fibre $\mathcal{C}_t$ a smooth and irreducible curve of genus g and special fibre $\mathcal{C}_0 = C$, then we can combinatorially compute g via the formula:

$$g = p_a(C) = h_1(G_C) + \sum_{i=1}^{v} g_i = e - v + 1 + \sum_{i=1}^{v} g_i.$$

Usually we will consider a curve C embedded in a projective space $\mathbb{P}^r$. In this situation each curve C_i will have a certain degree d_i, and we will consider the graph G_C as *double weighted*, by attaching to each vertex the pair of weights (g_i, d_i). Moreover we will attribute to the graph a further marking number, i.e. the embedding dimension r of C.

The total degree of C is

$$d = \sum_{i=1}^{v} d_i$$

which is also invariant by flat degeneration.

If each curve C_i is a line, the curve C is called a *stick curve*. In this case the double weighting is $(0, 1)$ for each vertex, and it will be omitted if no confusion arises.

It should be stressed that it is not true that for any simple, connected, double weighted graph G there is a curve C in a projective space such that $G_C = G$. For example there is no stick curve corresponding to the graph of Figure 1.

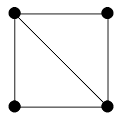
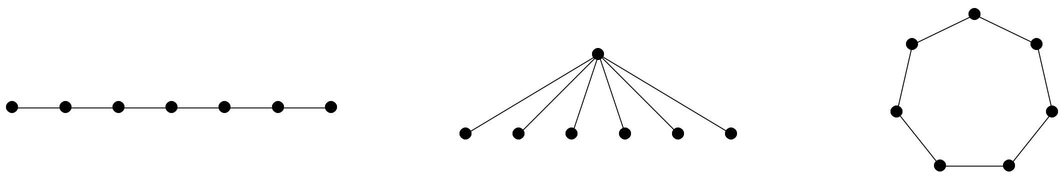


FIGURE 1. Dual graph of an "impossible" stick curve.

We now give two examples of stick curves which will be frequently used in this paper.

Example 2.8. Let T_n be any connected tree with $n \geqslant 3$ vertices. This corresponds to a non-degenerate stick curve of degree n in $\mathbb{P}^n$, which we denote by C_{T_n}. Indeed one can check that, taking a general point p_i on each component of C_{T_n}, the line bundle $\mathcal{O}_{C_{T_n}}(p_1 + \cdots + p_n)$ is very ample. Of course C_{T_n} has arithmetic genus 0 and is a flat limit of rational normal curves in $\mathbb{P}^n$.

We will often consider two particular trees T_n: a chain R_n of length n and the fork S_n with $n-1$ teeth, i.e. a tree consisting of $n-1$ vertices joining a further vertex (see Figures 2.(a) and (b)). The curve C_{R_n} is the union of n lines $l_1, l_2, \dots, l_n$ spanning $\mathbb{P}^n$, such that $l_i \cap l_j = \emptyset$ if and only if $1 < |i-j|$. The curve C_{S_n} is the union of n lines $l_1, l_2, \dots, l_n$ spanning $\mathbb{P}^n$, such that $l_1, \dots, l_{n-1}$ all intersect l_n at distinct points (see Figure 3).



(a) A chain R_n (b) A fork S_n with $n-1$ teeth (c) A cycle E_n

FIGURE 2. Examples of dual graphs.

Example 2.9. Let Z_n be any simple, connected graph with $n \geqslant 3$ vertices and $h^1(Z_n, \mathbb{C}) = 1$. This corresponds to a projectively normal stick curve of degree n in $\mathbb{P}^{n-1}$, which we denote by C_{Z_n} (as in Example 2.8). The curve C_{Z_n} has arithmetic genus 1 and it is a flat limit of elliptic normal curves in $\mathbb{P}^{n-1}$.

We will often consider the particular case of a cycle E_n of order n (see Figure 2.c). The curve C_{E_n} is the union of n lines $l_1, l_2, \dots, l_n$ spanning $\mathbb{P}^{n-1}$, such that $l_i \cap l_j = \emptyset$ if and only if $1 < |i-j| < n-1$ (see Figure 3).

We remark that C_{E_n} is projectively Gorenstein, because $\omega_{C_{E_n}}$ is trivial, since there is an everywhere non-zero global section of $\omega_{C_{E_n}}$, given by the meromorphic 1-form on each component with residues 1 and -1 at the nodes (in a suitable order).

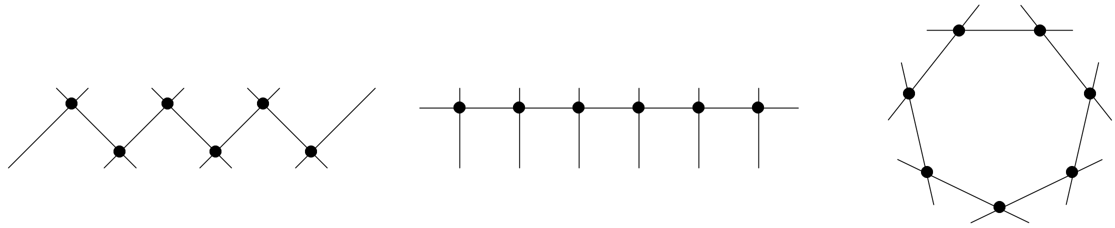
All the other C_{Z_n}'s, instead, are not locally Gorenstein because $\omega_{C_{Z_n}}$, although of degree zero, is not trivial. Indeed a graph Z_n, different from E_n, certainly has a vertex with valence 1. This corresponds to a line l such that $\omega_{C_{Z_n}} \otimes \mathcal{O}_l$ is not trivial.

3. ZAPPATIC SURFACES AND ASSOCIATED GRAPHS

First of all, we need to introduce the singularities we will allow.

Definition 3.1 (Zappatic singularity)**.** Let X be a surface and let $x \in X$ be a point. We will say that x is a *Zappatic singularity* for X if (X, x) is locally analytically isomorphic to a pair (Y, y) where Y is the cone over either a curve C_{T_n} or a curve C_{Z_n}, $n \geqslant 3$, and y is the vertex of the cone. Accordingly we will say that x is either a T_n- or a Z_n*-point* for X.

Definition 3.2 (Zappatic surface)**.** Let X be a projective surface with its irreducible components $X_1, \dots, X_v$. We will assume that X has the following properties:



C_{R_n}: a chain of n lines, C_{S_n}: a comb with $n-1$ teeth, C_{E_n}: a cycle of n lines.

FIGURE 3. Examples of stick curves.

- X is reduced and connected in codimension one;
- $X_1, \dots, X_v$ are smooth;
- the singularities in codimension one of X are at most double curves which are smooth and irreducible;
- the further singularities of X are Zappatic singularities.

A surface like X will be called a *Zappatic surface*. If moreover X is embedded in a projective space $\mathbb{P}^r$ and all of its irreducible components are planes, we will say that X is a *planar Zappatic surface*.

Notation 3.3. Let X be a Zappatic surface. Let us denote by:
- X_i : an irreducible component of X, $i \leqslant i \leqslant v$;
- $C_{ij} := X_i \cap X_j$, $1 \leqslant i \neq j \leqslant v$, if X_i and X_j meet along a curve, otherwise set $C_{ij} = \emptyset$;
- g_{ij} : the genus of C_{ij}, $1 \leqslant i \neq j \leqslant v$;
- $C := \mathrm{Sing}(X) = \cup_{i<j} C_{ij}$, the union of all the double curves of X;
- $\Sigma_{ijk} := X_i \cap X_j \cap X_k$, $1 \leqslant i \neq j \neq k \leqslant v$, if $X_i \cap X_j \cap X_k \neq \emptyset$, otherwise $\Sigma_{ijk} = \emptyset$;
- m_{ijk} : the cardinality of the set Σ_{ijk};
- P^h_{ijk} : the Zappatic singular point belonging to Σ_{ijk}, for $h = 1, \dots, m_{ijk}$.

Furthermore, if $X \subset \mathbb{P}^r$, for some r, we denote by
- d : the degree of X;
- d_i : the degree of X_i, $i \leqslant i \leqslant v$;
- c_{ij} : the degree of C_{ij}, $1 \leqslant i \neq j \leqslant v$;
- D : a general hyperplane section of X;
- g : the arithmetic genus of D;
- D_i : the (smooth) irreducible component of D lying in X_i, which is a general hyperplane section of X_i, $1 \leqslant i \leqslant v$;
- g_i : the genus of D_i, $1 \leqslant i \leqslant v$.

If moreover X is a planar Zappatic surface, then $d = v$, each non-empty set Σ_{ijk} is a singleton and $m_{ijk} = 1$, for each $i \neq j \neq k$.

Remark 3.4. A Zappatic surface X is locally Cohen-Macaulay. Thus the dualizing sheaf ω_X is well-defined. If X has only E_n-points as Zappatic singularities, then X is locally Gorenstein, hence ω_X is an invertible sheaf. If X has global normal crossings, i.e. if X has only E_3-points as Zappatic singularities, we define the *geometric genus* of X as:

$$p_g(X) := h^0(X, \omega_X). \tag{3.5}$$

If X is smoothable, namely if X is the central fibre of a degeneration, we will define its geometric genus later in Definition 5.18.

Definition 3.6 (Good Zappatic surface)**.** The *good Zappatic singularities* are the
- R_n-points, for $n \geqslant 3$,

- S_n-points, for $n \geqslant 4$,
- E_n-points, for $n \geqslant 3$,

which are the Zappatic singularities whose associated stick curves are respectively C_{R_n}, C_{S_n}, C_{E_n} (see Examples 2.8 and 2.9, Figures 2, 3 and 4).

A *good Zappatic surface* is a Zappatic surface with only good Zappatic singularities.

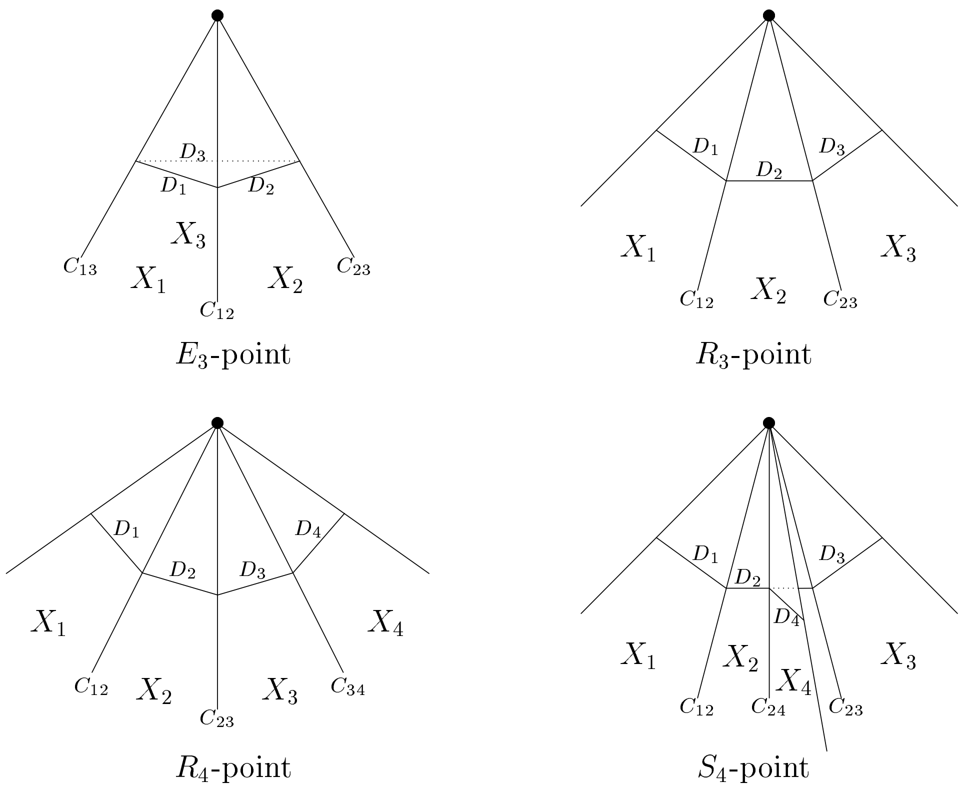


FIGURE 4. Examples of good Zappatic singularities.

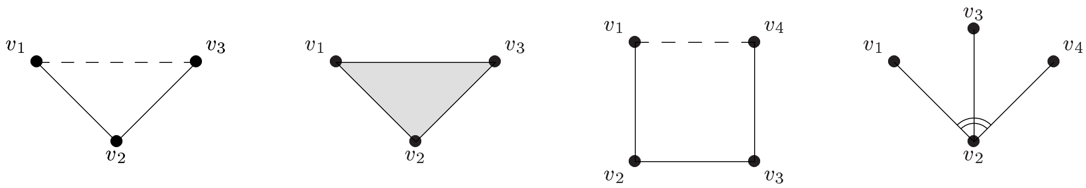
To a good Zappatic surface X we can associate a complex G_X, which we briefly call the *associated graph* to X.

Definition 3.7 (The associated graph to X)**.** Let X be a good Zappatic surface with Notation 3.3. The graph G_X associated to X is defined as follows (cf. Figure 5):

- each surface X_i corresponds to a vertex v_i;
- each double curve C_{ij} correspond to an edge e_{ij} joining v_i and v_j. The edge $e_{ij}, i < j$, is oriented from the vertex v_i to the one v_j;
- each E_n-point P of X is a face of the graph whose n edges correspond to the double curves concurring at P. This is called a *n-face* of the graph;
- for each R_n-point P, with $n \geqslant 3$, if $P \in X_{i_1} \cap X_{i_2} \cap \cdots \cap X_{i_n}$, where X_{i_j} meets X_{i_k} along a curve $C_{i_j i_k}$ only if $1 = |j-k|$, we add in the graph a *dashed edge* joining the vertices corresponding to X_{i_1} and X_{i_n}. The dashed edge e_{i_1,i_n}, together with the other $n-1$ edges $e_{i_j,i_{j+1}}$, $j = 1, \ldots, n-1$, bound an *open n-face* of the graph;
- for each S_n-point P, with $n \geqslant 4$, if $P \in X_{i_1} \cap X_{i_2} \cap \cdots \cap X_{i_n}$, where $X_{i_1}, \ldots, X_{i_{n-1}}$ all meet X_{i_n} along curves $C_{i_j i_n}$, $j = 1, \ldots, n-1$, concurring at P, we mark this in the graph by an a *angle* spanned by the edges corresponding to the curves $C_{i_j i_n}$, $j = 1, \ldots, n-1$.

In the sequel, when we speak of *faces* of G_X we always mean closed faces. Of course each vertex v_i is weighted with the relevant invariants of the corresponding surface X_i. We will usually omit these weights if X is planar, i.e. if all the X_i's are planes.

Since each R_n-, S_n-, E_n-point is an element of some set of points Σ_{ijk} (cf. Notation 3.3), we remark that there can be different faces (as well as open faces and angles) of G_X which are incident on the same set of vertices and edges. However this cannot occur if X is planar.



R_3-point E_3-point R_4-point S_4-point

FIGURE 5. Associated graphs of R_3-, E_3-, R_4- and S_4-points (cf. Figure 4).

Notice that angles, open and closed faces of G_X have been defined in order to encode the good Zappatic singularities of X. In other words, the associated graph G_X uniquely determines the configuration of the good Zappatic singularities of X.

Consider three vertices v_i, v_j, v_k of G_X in such a way that v_i is joint with v_j and v_k. Any point in $C_{ij} \cap C_{ik}$ is either a R_n-, or a S_n-, or an E_n-point, and the curves C_{ij} and C_{ik} intersect transversally, by definition of Zappatic singularities. Hence we can compute the intersection number $C_{ij} \cdot C_{ik}$ by adding the number of closed and open faces and of angles involving the edges e_{ij}, e_{ik}. In particular, if X is planar, for every pair of adjacent edges only one of the following possibilities occur: either they belong to an open face, or to a closed one, or to an angle. Therefore for good, planar Zappatic surfaces we can avoid to mark open 3-faces without loosing any information (see Figure 6, cf. Figure 5).

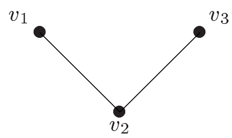


FIGURE 6. Associated graph of a R_3-point in a good, planar Zappatic surface.

Remark 3.8. We also notice that, given our choices, if X is good Zappatic and has only E_3-points, the graph G_X comes with a lexicographic orientation of the faces; indeed, let $\Sigma_{ijk} = X_i \cap X_j \cap X_k = \{P^1_{ijk}, P^2_{ijk}, \ldots, P^{m_{ijk}}_{ijk}\}$; thus, each face of G_X corresponds to a sequence of three vertices i, j, k with $i < j < k$, together with an integer t such that $1 \leqslant t \leqslant m_{ijk}$, hence it will be denoted by $f^t_{\sigma(i),\sigma(j),\sigma(k)}$, with σ any permutation of i, j, k, and will be oriented according to the orientation of its boundary determined by the sequence of vertices v_i, v_j, v_k.

As for stick curves, if G is a given graph as above, there does not necessarily exist a good planar Zappatic surface X such that its associated graph is $G = G_X$.

Example 3.9. Consider the graph G of Figure 7. If G were the associated graph of a good planar Zappatic surface X, then X should be a global normal crossing union of 4 planes with 5 double lines and two E_3 points, P_{123} and P_{134}, both lying on the double line C_{13}. Since the lines C_{23} and C_{34} (resp. C_{14} and C_{12}) both lie on the plane X_3 (resp. X_1), they should intersect. This means that the planes X_2, X_4 also should intersect along a line, therefore the edge e_{24} should appear in the graph.

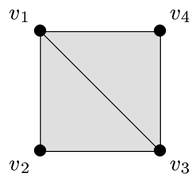


FIGURE 7. Graph associated to an impossible planar Zappatic surface.

Before going on, we need some notation.

Notation 3.10. Let X be a good Zappatic surface (with invariants as in Notation 3.3) and let $G = G_X$ be its associated graph. We denote by

- v : the number of vertices of G (i.e. the number of irreducible components of X);
- V : the (indexed) set of vertices of G;
- e : the number of edges of G (i.e. the number of double curves in X);
- E : the set of edges of G; this is indexed by the ordered pairs $(i,j) \in V \times V$, $i < j$, such that the corresponding surfaces X_i, X_j meet along the curve $C_{ij} = C_{ji}$;
- f_n : the number of n-faces of G, i.e. the number of E_n-points of X, for $n \geqslant 3$;
- $f := \sum_{n \geqslant 3} f_n$, the number of faces of G, i.e. the total number of E_n-points of X, for all $n \geqslant 3$;
- r_n : the number of open n-faces of G, i.e. the number of R_n-points of X, for $n \geqslant 3$;
- s_n : the number of n-angles of G, i.e. the number of S_n-points of X, for $n \geqslant 4$;
- $\chi(G) := v - e + f$, i.e. the Euler-Poincaré characteristic of G;
- $G^{(1)}$: the 1-*skeleton* of G, i.e. the graph obtained from G by forgetting all the faces, dashed edges and angles;
- $\chi(G^{(1)}) = v - e$, i.e. the Euler-Poincaré characteristic of $G^{(1)}$.

Remark 3.11. Observe that, when X is a good, planar Zappatic surface, the 1-skeleton $G_X^{(1)}$ of G_X coincides with the dual graph G_D of the general hyperplane section D of X.

Now we can compute some of the invariants of good Zappatic surfaces.

Proposition 3.12. *Let $X = \bigcup_{i=1}^{v} X_i \subset \mathbb{P}^r$ be a good Zappatic surface and let $G = G_X$ be its associated graph. Let C be the double locus of X, i.e. the union of the double curves of X, $C_{ij} = C_{ji} = X_i \cap X_j$ and let $c_{ij} = \deg(C_{ij})$. Let D_i be a general hyperplane section of X_i, and denote by g_i its genus. Then the arithmetic genus of a general hyperplane section D of X is:*

$$g = \sum_{i=1}^{v} g_i + \sum_{e_{ij} \in E} c_{ij} - v + 1. \tag{3.13}$$

In particular, when X is a good, planar Zappatic surface, then

$$g = e - v + 1 = 1 - \chi(G^{(1)}). \tag{3.14}$$

Proof. Denote by d_i the degree of X_i, $1 \leqslant i \leqslant v$. Then, D is the union of the v irreducible components D_i, $1 \leqslant i \leqslant v$, such that $\deg(D_i) = d_i$ and $d := \deg(D) = \sum_{i=1}^{v} d_i$. Consider its associated graph G_D, defined as in §2.

Take G, whose indexed set of edges is denoted by E, and consider $e_{ij} \in E$ joining its vertices v_i and v_j, $i < j$, which correspond to the irreducible components X_i and X_j, respectively. Since e_{ij} in G correspond to the double curve C_{ij}, we have exactly c_{ij} oriented edges in the graph G_D joining its vertices v_i and v_j, which now correspond to the irreducible components of D, D_i and D_j, respectively. These c_{ij} oriented edges correspond to the c_{ij} nodes of the reducible curve $D_i \cup D_j$, which is part of the hyperplane section D.

Now, recall that the Hilbert polynomial of D is, with our notation, $P_D(t) = dt + 1 - g$. On the other hand, $P_D(t)$ equals the number of independent conditions imposed on hypersurfaces $\mathcal{H}$ of degree $t \gg 0$ to contain D.

From what observed above on G_D, it follows that the number of singular points of D is $\sum_{e_{ij} \in E} c_{ij}$. These points impose independent conditions on hypersurfaces $\mathcal{H}$ of degree $t \gg 0$.

Since $t \gg 0$ by assumption, we get that the map

$$H^0(\mathcal{O}_{\mathbb{P}^r}(t)) \to H^0(\mathcal{O}_{D_i}(t))$$

is surjective and that the line bundle $\mathcal{O}_{D_i}(t)$ is non-special on D_i, for each $1 \leqslant i \leqslant v$. Thus, in order for $\mathcal{H}$ to contain D_i we have to impose $d_i t - g_i + 1 - \sum_{j \text{ s.t. } e_{ij} \in E} c_{ij}$ conditions. Therefore the total number of conditions for $\mathcal{H}$ to contain D is:

$$\sum_{e_{ij} \in E} c_{ij} + \sum_{i=1}^{v} \Big(d_i t - g_i + 1 - \sum_{j, e_{ij} \in E} c_{ij} \Big) = \sum_{e_{ij} \in E} c_{ij} + dt - \sum_{i=1}^{v} g_i + v - \sum_{i=1}^{v} \sum_{j, e_{ij} \in E} c_{ij} =$$

$$= dt + v - \sum_{i=1}^{v} g_i - \sum_{e_{ij} \in E} c_{ij},$$

since $\sum_{i=1}^{v} \sum_{j, e_{ij} \in E} c_{ij} = 2 \sum_{e_{ij} \in E} c_{ij}$. This proves (3.13) (cf. formula (2.4)).

The second part of the statement directly follows from the above computations and from the fact that, in the good planar Zappatic case $g_i = 0$ and $c_{ij} = 1$, for each $i < j$, i.e. G_D coincides with $G^{(1)}$ (cf. Remark 3.11). □

By recalling Notation 3.10, one also has:

Proposition 3.15. *Let $X = \bigcup_{i=1}^{v} X_i$ be a good Zappatic surface and G_X be its associated graph. Let C be the double locus of X, which is the union of the curves $C_{ij} = C_{ji} = X_i \cap X_j$. Then:*

$$\chi(\mathcal{O}_X) = \sum_{i=1}^{v} \chi(\mathcal{O}_{X_i}) - \sum_{e_{ij} \in E} \chi(\mathcal{O}_{C_{ij}}) + f. \tag{3.16}$$

In particular, when X is a good, planar Zappatic surface, then

$$\chi(\mathcal{O}_X) = \chi(G_X) = v - e + f. \tag{3.17}$$

Proof. We can consider the sheaf morphism:

$$\bigoplus_{i=1}^{v} \mathcal{O}_{X_i} \xrightarrow{\lambda} \bigoplus_{1 \leqslant i < j \leqslant v} \mathcal{O}_{C_{ij}}, \tag{3.18}$$

defined in the following way: if

$$\pi_{ij} : \bigoplus_{1 \leqslant i < j \leqslant v} \mathcal{O}_{C_{ij}} \to \mathcal{O}_{C_{ij}}$$

denotes the projection on the $(ij)^{\text{th}}$-summand, then

$$(\pi_{ij} \circ \lambda)(h_1, \ldots, h_v) := h_i - h_j.$$

Notice that the definition of λ is consistent with the lexicographic order of the indices and with the lexicographic orientation of the edges of the graph G_X.

Observe that, if $\tilde{X}$ denotes the desingularization of X, then $\tilde{X}$ is isomorphic to the disjoint union of the smooth, irreducible components X_i, $1 \leqslant i \leqslant v$, of X. Therefore, by the very definition of $\mathcal{O}_X$, we see that

$$\ker(\lambda) \cong \mathcal{O}_X.$$

We claim that the morphism λ is not surjective and that its cokernel is a sky-scraper sheaf supported at the E_n-points of X. To show this, we focus on any irreducible component of $C = \bigcup_{1 \leqslant i < j \leqslant v} C_{ij}$, the double locus of X.

Fix any index pair (i, j), with $i < j$, and consider the generator

$$(0, \dots, 0, 1, 0, \dots, 0) \in \bigoplus_{1 \leqslant l < m \leqslant v} \mathcal{O}_{C_{lm}}, \tag{3.19}$$

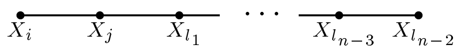
where $1 \in \mathcal{O}_{C_{ij}}$, the $(ij)^{\text{th}}$-summand. The obstructions to lift up this element to an element of $\bigoplus_{1 \leqslant t \leqslant v} \mathcal{O}_{X_t}$ are given by the presence of good Zappatic singularities of X along C_{ij}.

For what concerns the irreducible components of X which are not involved in the intersection determining a good Zappatic singularity on C_{ij}, the element in (3.19) trivially lifts-up to 0 on each of them. Thus, in the sequel, we shall focus only on the irreducible components involved in the Zappatic singularity, which will be denoted by $X_i,\ X_j,\ X_{l_t}$, for $1 \leqslant t \leqslant n-2$.

We have to consider different cases, according to the good Zappatic singularity type lying on the curve $C_{ij} = X_i \cap X_j$.

- Suppose that C_{ij} passes through a R_n-point P of X, for some n; we have two different possibilities. Indeed:

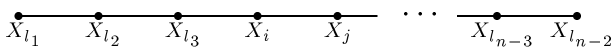
  (a) let X_i be an "external" surface for P — i.e. X_i corresponds to a vertex of the associated graph of P which has valence 1. Therefore, we have:

  In this situation, the element in (3.19) lifts up to

  $$(1, 0, \dots, 0) \in \mathcal{O}_{X_i} \oplus \mathcal{O}_{X_j} \oplus \bigoplus_{1 \leqslant t \leqslant n-2} \mathcal{O}_{X_{l_t}}.$$

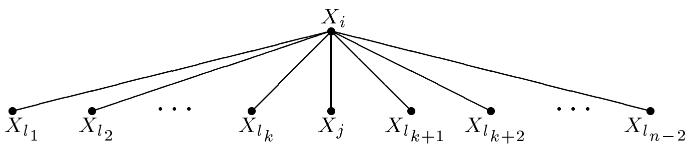
  (b) let X_i be an "internal" surface for P — i.e. X_i corresponds to a vertex of the associated graph to P which has valence 2. Thus, we have a picture like:

  In this case, the element in (3.19) lifts up to the n-tuple having components:
  $1 \in \mathcal{O}_{X_i}$,
  $0 \in \mathcal{O}_{X_j}$,
  $1 \in \mathcal{O}_{X_{l_t}}$, for those X_{l_t}'s corresponding to vertices in the graph associated to P which are on the left of X_i and,
  $0 \in \mathcal{O}_{X_{l_k}}$ for those X_{l_k}'s corresponding to vertices in the graph associated to P which are on the right of X_j.

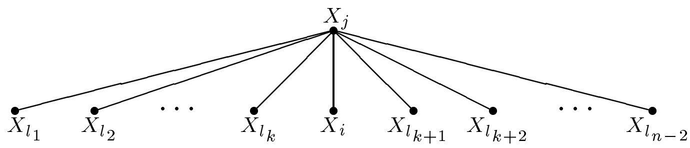
- Suppose that C_{ij} passes through a S_n-point P of X, for any n; as before, we have two different possibilities. Indeed:

  (a) let X_i corresponds to the vertex of valence $n-1$ in the associated graph to P, i.e.

  In this situation, the element in (3.19) lifts up to the n-tuple having components:
  $1 \in \mathcal{O}_{X_i}$,
  $0 \in \mathcal{O}_{X_j}$,
  $1 \in \mathcal{O}_{X_{l_t}}$, for all $1 \leqslant t \leqslant n-2$.

(b) let X_i corresponds to a vertex of valence 1 in the associated graph to P. Since $C_{ij} \neq \emptyset$ by assumption, then X_j has to be the vertex of valence $n-1$, i.e. we have the following picture:



Thus, the element in (3.19) lifts up to the n-tuple having components
$1 \in \mathcal{O}_{X_i}$,
$0 \in \mathcal{O}_{X_j}$,
$0 \in \mathcal{O}_{X_{l_t}}$, for all $1 \leqslant t \leqslant n-2$.

- Suppose that C_{ij} passes through an E_n-point P for X. Then, each vertex of the associated graph to P has valence 2. Since such a graph is a cycle, it is clear that no lifting of (3.19) can be done.

To sum up, we see that $\mathrm{coker}(\lambda)$ is supported at the E_n-points of X. Furthermore, if we consider

$$\bigoplus_{1 \leqslant i < j \leqslant v} \mathcal{O}_{C_{ij}} \xrightarrow{ev_P} \mathcal{O}_P = \underline{\mathbb{C}}_P, \qquad \oplus f_{ij} \mapsto \sum f_{ij}(P)$$

it is clear that, if P is an E_n-point then

$$ev_P \left(\bigoplus_{1 \leqslant i < j \leqslant v} \mathcal{O}_{C_{ij}} / \operatorname{Im}(\lambda) \right) \cong \underline{\mathbb{C}}_P.$$

This means that

$$\mathrm{coker}(\lambda) \cong \underline{\mathbb{C}}^f.$$

By the exact sequences

$$0 \to \mathcal{O}_X \to \bigoplus_{1 \leqslant i \leqslant v} \mathcal{O}_{X_i} \to \operatorname{Im}(\lambda) \to 0, \qquad 0 \to \operatorname{Im}(\lambda) \to \bigoplus_{1 \leqslant i < j \leqslant v} \mathcal{O}_{C_{ij}} \to \underline{\mathbb{C}}^f \to 0,$$

we get (3.16). □

In the next section we will see how the computation of the geometric genus is much more involved, even in the case of X having only E_3-points as Zappatic singularities.

4. The geometric genus of a Zappatic surface with only E_3-points

The main purpose of this section is to compute the *geometric genus* of a projective good Zappatic surface $X = \bigcup_{i=1}^{v} X_i$ (cf. Remark 3.4), which is assumed to have only E_3-points, i.e. global normal crossing singularities. In terms of its associated graph, this means that G_X is a subgraph of the complete graph on v vertices, which has only 3-faces (i.e. triangles).

We first want to recall some definitions and results, which will be used in the sequel.

Definition 4.1. Let T be a smooth surface and C be a smooth, irreducible curve on T. Let ω be a global meromorphic 2-form on T whose polar locus contains C. We may assume that x, y are local coordinates on T in an analytic neighbourhood of a point of C in such a way that $y = 0$ is the local equation defining C. Then, the *Poincaré residue map* (or *adjunction map*)

$$\omega_T \otimes \mathcal{O}_T(C) \xrightarrow{\mathcal{R}_C} \omega_C$$

is locally defined by:

$$\omega = \frac{f(x,y)}{y}\, dx \wedge dy \ \mapsto \ -f(x,0)\, dx$$

and $-f(x,0)\,dx$ is called the *(Poincaré) residue* of the 2-form ω along C (see [10, p. 147]).

If, more generally, C is assumed to be a reduced (possibly reducible) divisor with only normal crossing singularities, denote by ω_C its dualizing sheaf and take local coordinates on T in an analytic neighbourhood of a node of C in such a way that $xy = 0$ is its local defining equation. If

$$\omega = \frac{f(x,y)}{xy}\, dx \wedge dy,$$

then one defines the Poincaré residue map by considering

$$\mathcal{R}_C(\omega) \in H^0(C, \omega_C) \tag{4.2}$$

defined as the pair of forms:

(i) $\omega_x = -\dfrac{f(x,0)}{x}\, dx$ on the branch $y = 0$,

(ii) $\omega_y = \dfrac{f(0,y)}{y}\, dy$ on the branch $x = 0$.

Remark 4.3. If $\mathcal{R}_0(\omega_x)$ denotes the usual Poincaré residue at the point $x = 0$ of the meromorphic 1-form ω_x on the (smooth) branch $y = 0$, observe that

$$\mathcal{R}_0(\omega_y) = -\mathcal{R}_0(\omega_x). \tag{4.4}$$

This is consistent with the definition of $H^0(C, \omega_C)$. Indeed, assume that C has only m nodes; then, if $\nu : \tilde{C} \to C$ is the normalization morphism and if $\{q_1, q_1'\}$, $\{q_2, q_2'\}, \ldots,$ $\{q_m, q_m'\}$ are the pre-images in $\tilde{C}$ of the m nodes of C, then ω_C is the invertible subsheaf

$$\omega_C \subset \nu_*(\omega_{\tilde{C}}(\sum_{i=1}^{m}(q_i + q_i')))$$

such that a section σ of $\nu_*(\omega_{\tilde{C}}(\sum_{i=1}^{m}(q_i + q_i')))$, viewed as a section of $\omega_{\tilde{C}}(\sum_{i=1}^{m}(q_i + q_i'))$, is a section of ω_C if and only if

$$\mathcal{R}_{q_i}(\sigma) + \mathcal{R}_{q_i'}(\sigma) = 0, \quad 1 \leqslant i \leqslant m.$$

Unless otherwise stated, from now on $X = \bigcup_{i=1}^{v} X_i$ will denote a projective, Zappatic surface with E_3-points only as Zappatic singularities and we use notation as in Definition 3.2 and in Notation 3.10.

It is well known that, for each i:

$$\omega_{X_i} \cong \omega_X \otimes \mathcal{O}_{X_i}(-C_i), \qquad \text{with } C_i := X_i \cap \overline{(X \setminus X_i)} = \sum_{j \neq i} C_{ij}, \tag{4.5}$$

where ω_{X_i} is the canonical line bundle of X_i, whereas ω_X denotes the dualizing sheaf of X. Note that ω_X is an invertible sheaf by the hypotheses on X.

As in Remark 3.4, recall that the geometric genus of X is denoted by $p_g(X)$ and defined as $p_g(X) = h^0(X, \omega_X)$. In order to compute $p_g(X)$, we need some further remarks which will be fundamental in the sequel.

Remark 4.6. Observe that, if $X = \bigcup_{i=1}^{v} X_i$ is as above, since the intersection $X_i \cap X_j$ — when non-empty — is the double curve $C_{ij} = C_{ji}$, the index pair (i,j) with $i < j$ uniquely determines the double curve C_{ij}. In the same way, when $i < j$, since the intersection $X_i \cap X_j \cap X_k$ — when non-empty — is the triple point set $\Sigma_{ijk} = \Sigma_{\tau(i)\tau(j)\tau(k)}$, for $k \neq i,j$ and for any $\tau \in \mathrm{Sym}(\{i,j,k\})$, then the lexicographically ordered index triple uniquely determines the triple point set either Σ_{kij}, or Σ_{ikj}, or Σ_{ijk}, according to the fact that either $k < i$, or $i < k < j$, or $k > j$.

Remark 4.7. Since the only Zappatic singularities of X are assumed to be E_3-points, then G_X contains neither dashed edges, nor angles, nor open faces, nor n-faces, with $n \geqslant 4$. Furthermore, in such a case the graph G_X comes with a lexicographic orientation of the faces (see Remark 3.8). If X is, in particular, planar recall that we have strong constraints on the possible shape of the graph G_X — because of the geometry of planes (cf. Example 3.9)— and each non-zero m_{ijk} equals one.

Observe that by the connectedness hypothesis of $X = \bigcup_{i=1}^{v} X_i$, we get that $C_i \neq \emptyset$, for each $1 \leqslant i \leqslant v$. For simplicity of notation, in the sequel we shall always denote by C_{ij} the intersection of X_i and X_j, for any $1 \leqslant i < j \leqslant v$, with the obvious further condition that $C_{ij} = C_{ji} = \emptyset$ when the index pair corresponds to two disjoint surfaces in X, i.e. when there is no edge (v_i, v_j) in the associated graph G_X.

We can define a natural map

$$\Phi : \bigoplus_{i=1}^{v} H^1(X_i, \mathcal{O}_{X_i}) \to \bigoplus_{1\leqslant i<j\leqslant v} H^1(C_{ij}, \mathcal{O}_{C_{ij}}) \tag{4.8}$$

in the following way: if

$$\pi_{ij} : \bigoplus_{1\leqslant i<j\leqslant v} H^1(C_{ij}, \mathcal{O}_{C_{ij}}) \to H^1(C_{ij}, \mathcal{O}_{C_{ij}})$$

denotes the projection on the $(ij)^{\text{th}}$-summand and if $r^{(i)}_{C_{ij}} : H^1(X_i, \mathcal{O}_{X_i}) \to H^1(C_{ij}, \mathcal{O}_{C_{ij}})$ denotes the natural restriction map to C_{ij} as a divisor in X_i, where $i < j$, then

$$(\pi_{ij} \circ \Phi)((a_1, \ldots, a_v)) := r^{(i)}_{C_{ij}}(a_i) - r^{(j)}_{C_{ij}}(a_j). \tag{4.9}$$

Remark 4.10. Observe that the above definition is consistent with the lexicographic order of the indices $1 \leqslant i \leqslant v$. In other words, (4.9) means that we consider the curve C_{ij} as a *positive curve* on the surface X_i and as a *negative curve* on the surface X_j, when $i < j$. Furthermore, when the index pair is such that $C_{ij} = \emptyset$, we obviously consider $H^1(C_{ij}, \mathcal{O}_{C_{ij}})$ as the zero-vector space and $\pi_{ij} \circ \Phi$ as the zero-map.

Take an index pair $i < j$ such that $C_{ij} \neq \emptyset$. By the adjunction sequence of C_{ij} on X_i and on X_j, we can consider the two obvious coboundary maps:

$$\begin{tikzcd}[row sep=small] & H^1(X_i, \omega_{X_i}) \\ H^0(C_{ij}, \omega_{C_{ij}}) \arrow[ur, "\delta_i"] \arrow[dr, "\delta_j"'] & \\ & H^1(X_j, \omega_{X_j}). \end{tikzcd} \tag{4.11}$$

On the other hand, when the index pair $i < j$ is such that $C_{ij} = \emptyset$, then $H^0(C_{ij}, \omega_{C_{ij}})$ is considered as the zero-vector space and (4.11) are the zero-maps. Then, we can define the map

$$\Delta : \bigoplus_{1\leqslant i<j\leqslant v} H^0(C_{ij}, \omega_{C_{ij}}) \to \bigoplus_{i=1}^{v} H^1(X_i, \omega_{X_i}) \tag{4.12}$$

in the following way: if

$$\iota_{ij} : H^0(C_{ij}, \omega_{C_{ij}}) \hookrightarrow \bigoplus_{1\leqslant i<j\leqslant v} H^0(C_{ij}, \omega_{C_{ij}})$$

denotes the natural inclusion of the $(ij)^{\text{th}}$-summand and if $\gamma_{ij} \in H^0(C_{ij}, \omega_{C_{ij}})$, then

$$(\Delta \circ \iota_{ij})(\gamma_{ij}) := (0, \ldots, 0, \delta_i(\gamma_{ij}), 0, \ldots, 0, -\delta_j(\gamma_{ij}), 0, \ldots, 0), \tag{4.13}$$

where $i < j$.

Observe that the definition of Δ is consistent with the lexicographic order of the indices $1 \leqslant i \leqslant v$ and with our Remark 4.10.

The following preliminary result is an obvious consequence of our definitions.

Proposition 4.14. *With notation as above, we have*

$$\Delta = \Phi^{\vee}.$$

Proof. The proof directly follows from Serre's duality on each summand and from the fact that the matrix which represents Δ is the transpose of the one representing Φ. □

We are now able to prove the main result of this section.

Theorem 4.15. *Let $X = \bigcup_{i=1}^{v} X_i$ be a projective, good Zappatic surface with only E_3-points as Zappatic singularities. Denote by ω_X the dualizing sheaf of X and by G_X the associated graph of X (see Definition 3.7). Let Φ be the map defined in* (4.8) *and let $p_g(X)$ be the geometric genus of X as in Remark 3.4. Then the following inequality:*

$$p_g(X) \leqslant b_2(G_X) + \sum_{i=1}^{v} p_g(X_i) + \dim(\mathrm{coker}(\Phi)) \tag{4.16}$$

holds, where as costumary $b_2(G_X)$ is the second Betti number of G_X.

Furthermore, a sufficient condition for the equality in (4.16) *to hold is either that*

(i) *each irreducible component X_i is a regular surface, for $1 \leqslant i \leqslant v$, or that*

(ii) *for any irregular component X_j of X, the divisor $C_j = X_j \cap \overline{(X \setminus X_j)}$ is ample on X_j.*

Proof. To prove the first part of the statement, we construct a homomorphism

$$H^0(\omega_X) \xrightarrow{f} H_2(G_X, \mathbb{C}) \tag{4.17}$$

and we show that

$$\ker(f) \cong \bigoplus_{i=1}^{v} H^0(X_i, \omega_{X_i}) \oplus (\mathrm{coker}(\Phi)). \tag{4.18}$$

Then, for what concerns the second part, we prove that either hypothesis (i) or hypothesis (ii) implies the surjectivity of the map (4.17).

Recall that, from our hypotheses it follows that G_X is a subgraph of the complete graph on d vertices which contains only 3-faces and that there can be more than one 3-face incident on the same triple of vertices (equiv. edges). This occurs when (cf. Notation 3.3) $m_{ijk} > 1$, for a given triple v_i, v_j, v_k of vertices of G_X. It is obvious that, when $\Sigma_{ijk} = \emptyset$, then $m_{ijk} = 0$.

To construct f, by (4.5), we consider a global section $\omega \in H^0(X, \omega_X)$ as a collection

$$\{\omega_i\}_{1 \leqslant i \leqslant v} \in \bigoplus_{1 \leqslant i \leqslant v} H^0(X_i, \omega_{X_i}(C_i)),$$

where each ω_i is a global meromorphic 2-form on the corresponding irreducible component X_i having simple polar locus along the (possibly reducible) curve

$$C_i = X_i \cap \overline{(X \setminus X_i)} = \sum_{j \neq i} C_{ij},$$

for each $1 \leqslant i \leqslant v$ (recall that $C_{ij} = C_{ji}$ and that $C_i \neq \emptyset$ for each $1 \leqslant i \leqslant v$, because of the connectedness hypothesis of X).

Take an index pair $i < j$ such that $C_{ij} \neq \emptyset$ and consider C_{ij}, which is both an irreducible component of C_i and of C_j. As in (4.2), take $\mathcal{R}_{C_{ij}}(\omega_i)$ the Poincaré residue of ω_i on C_{ij} and denote it by ω_{ij}. Then, we have

$$\omega_{ij} = -\omega_{ji}, \tag{4.19}$$

for each $1 \leqslant i < j \leqslant v$. In the trivial case $C_{ij} = \emptyset$, we have $\omega_{ij} = \omega_{ji} = 0$; so (4.19) holds.

Fix an index pair $i < j$ such that $C_{ij} \neq \emptyset$. By recalling our Remark 4.6 and by our hypotheses, when $X_i \cap X_j \cap X_k \neq \emptyset$ the lexicographically ordered index triple uniquely determine the triple point set either Σ_{kij}, or Σ_{ikj}, or Σ_{ijk} on C_{ij}, according to the case that either $k < i$, or $i < k < j$, or $k > j$. Now, given $i < j$ as above, we can consider ω_{ij} which is a meromorphic 1-form on the curve $C_{ij} \subset X_i$ having simple poles at the points in $\Sigma_{kij}, \Sigma_{ikj}, \Sigma_{ijk} \subset C_{ij}$ defined above and determined by those $k \neq i, j$ such that $X_i \cap X_j \cap X_k \neq \emptyset$. Otherwise, when k is such that $\Sigma_{ijk} = \emptyset$, then ω_{ij} must be considered as holomorphic at Σ_{ijk} and its residues at the empty point set are zero; such k's are determined by those vertices v_k in G_X which do not form any 3-face with the edge $e_{ij} = (v_i, v_j)$.

On the other hand, if the index pair $i < j$ is such that $C_{ij} = \emptyset$, one has that $\Sigma_{kij} = \Sigma_{ikj} = \Sigma_{ijk} = \emptyset$ for each $k \neq i, j$ and that each ω_{ij} is zero.

Therefore, for simplicity of notation, for any index pair $i < j$ we write

$$\omega_{ij} \in H^0\Big(C_{ij}, \omega_{C_{ij}}\Big(\sum_{k<i} \Sigma_{kij} + \sum_{k\in(i,j)} \Sigma_{ikj} + \sum_{k>j} \Sigma_{ijk}\Big)\Big),$$

recalling that, if $C_{ij} = \emptyset$, the above is the zero-vector space otherwise, if $C_{ij} \neq \emptyset$ but some triple point set is the empty set, its points do not impose any pole to the meromorphic form ω_{ij}.

In any case, one can compute the Poincaré residues at the given points, namely $\mathcal{R}_{P^r_{kij}}(\omega_{ij})$, $\mathcal{R}_{P^s_{ikj}}(\omega_{ij})$ and $\mathcal{R}_{P^t_{ijk}}(\omega_{ij})$, for any $1 \leqslant r \leqslant m_{kij}$, $1 \leqslant s \leqslant m_{ikj}$ and $1 \leqslant t \leqslant m_{ijk}$.

To simplify our notation, if e.g. $k > j$, we write $(\omega^t_{ij})_k$ (or directly ω^t_{ijk}) to denote the Poincaré residue $\mathcal{R}_{P^t_{ijk}}(\omega_{ij})$ of ω_{ij} at the t^{th} point P^t_{ijk} of the set Σ_{ijk}, for any $1 \leqslant t \leqslant m_{ijk}$. Similar notation for the other two cases.

As observed in Remark 4.6, if we fix $i < j < k$ we focus on the triple point set Σ_{ijk} of X, which is given by

$$X_i \cap X_j \cap X_k = C_{ij} \cap C_{ik} \cap C_{jk},$$

where

$$C_{ij}, C_{ik} \subset X_i, \quad C_{ji} = C_{ij}, C_{jk} \subset X_j, \quad C_{ki} = C_{ik}, C_{kj} = C_{jk} \subset X_k.$$

Therefore, at any given triple point P^t_{ijk}, with $i < j < k$ and $1 \leqslant t \leqslant m_{ijk}$, one can compute six different residues. Indeed, once we choose one of the three surfaces as the ambient variety, we have two different possible choices of smooth, irreducible curves (and so of meromorphic 1-forms) to use for such a computation. By taking into account (4.19), Remark 4.3 and the lexicographic order of the indices, the residues at P^t_{ijk} satisfy

$$\omega^t_{\sigma(i)\sigma(j)\sigma(k)} = \mathrm{sgn}(\sigma)\, \omega^t_{ijk}, \tag{4.20}$$

where $i < j < k$, $1 \leqslant t \leqslant m_{ijk}$ and where $\sigma \in \mathrm{Sym}(\{i, j, k\})$. Therefore, for each section $\omega \in H^0(X, \omega_X)$ there is, up to sign, a well determined value associated to each point $P^t_{ijk} \in \Sigma_{ijk}$. Recall that each such value is zero either if Σ_{ijk} is the empty point set or if some of the double curves is the empty set.

For $i < j$ such that $C_{ij} \neq \emptyset$, the subsets of triple points of X lying on the curve C_{ij} are parametrized by those indices $k \neq i, j$ such that the vertex v_k of G_X forms a number bigger than or equal to one of faces with the edge $e_{ij} = (v_i, v_j)$. By the Residue theorem on C_{ij} and by (4.20), we get

$$\sum_{k \neq i,j} \sum_{t=1}^{m_{\sigma_k(i)\sigma_k(j)\sigma_k(k)}} \operatorname{sgn}(\sigma_k)\, \omega^t_{\sigma_k(i)\sigma_k(j)\sigma_k(k)} = 0, \tag{4.21}$$

where $\sigma_k \in \operatorname{Sym}(\{i,j,k\})$ is the permutation which lexicographically reorders the index triples (i,j,k).

Otherwise, if $C_{ij} = \emptyset$, then (4.21) is trivially true, i.e. a sum of zeroes equals zero.

Choose, once and for all, the lexicographic orientation on the graph G_X. Therefore, if the edge $\lambda_{ij} = (v_i, v_j)$ belongs to G_X, then it is an arrow from v_i to v_j if and only if $i < j$. Recall that a set Σ_{ijk}, for some $i < j < k$, corresponds to $m_{ijk} = |\Sigma_{ijk}|$ faces of G_X insisting on the triple of vertices $\{v_i, v_j, v_k\}$. For any $1 \leqslant t \leqslant m_{ijk}$, we associate to the t^{th}-face the residue ω^t_{ijk} computed as above.

From (4.20), (4.21) and from the fact that G_X is a 2-dimensional graph, it follows that the above computations determine a 2-cycle $\{\omega^t_{ijk}\}$ of the graph G_X.

To sum up, the map f is defined as a composition of maps in the following way:

$$\begin{aligned} H^0(\omega_X) &\overset{i}{\hookrightarrow} \oplus H^0(\omega_{X_i}(C_i)) \xrightarrow{a} \oplus H^0(\omega_{C_i}) \xrightarrow{b} \\ &\xrightarrow{b} \oplus H^0(\omega_{C_{ij}}(\sum_{k,t} P^t_{ijk})) \xrightarrow{c} \oplus H^0(\omega_{C_{ij}}(\sum_{k,t} P^t_{ijk}))_{|P^t_{ijk}}) \to H_2(G_X, \mathbb{C}) \end{aligned}$$

where i is the natural inclusion and a, b, c are given by the following exact sequences:

$$0 \to \oplus H^0(\omega_{X_i}) \to \oplus H^0(\omega_{X_i}(C_i)) \xrightarrow{a} \oplus H^0(\omega_{C_i})$$

$$0 \to \oplus H^0(\omega_{C_i}(-C_{ij})) \to \oplus H^0(\omega_{C_i}) \xrightarrow{b} \oplus H^0(\omega_{C_i|C_{ij}}) \cong \oplus H^0(\omega_{C_{ij}}(\sum_{k,t} P^t_{ijk}))$$

$$0 \to \oplus H^0(\omega_{C_{ij}}(\sum_{l \neq k, r \neq t} P^r_{ijl})) \to \oplus H^0(\omega_{C_{ij}}(\sum_{l,r} P^r_{ijl})) \xrightarrow{c} \oplus H^0(\omega_{C_{ij}}(\sum_{l,r} P^r_{ijl}))_{|P^t_{ijk}})$$

To compute $\ker(f)$, take as above $C_i = X_i \cap \overline{(X \setminus X_i)} = \sum_{j \neq i} C_{ij}$, where we recall that $C_{ij} = C_{ji}$, for $i \neq j$, some — but not all — of them possibly empty. As in formulas (4.2) and (4.11), denote by $\mathcal{R}_{C_i}$ the Poincaré residue map on the reducible, nodal curve $C_i \subset X_i$ and by δ_i the coboundary map on the surface X_i, for $1 \leqslant i \leqslant v$. Thus, the first row of the diagram:

$$\begin{tikzcd} 0 \arrow[r] & \bigoplus_{i=1}^{v} H^0(\omega_{X_i}) \arrow[r] \arrow[dr, "\varsigma"'] & \bigoplus_{i=1}^{v} H^0(\omega_{X_i}(C_i)) \arrow[r, "\oplus \mathcal{R}_{C_i}"] & \bigoplus_{i=1}^{v} H^0(\omega_{C_i}) \arrow[r, "\oplus \delta_i"] & \bigoplus_{i=1}^{v} H^1(\omega_{X_i}) \\ & & \ker(f) \arrow[u, "\iota"] \arrow[r, "\sigma_{1/2}"] & \bigoplus_{i<j} H^0(\omega_{C_{ij}}) \arrow[u, "\beta"] \arrow[ur, "\Delta"'] & \end{tikzcd} \tag{4.22}$$

is naturally defined and exact. Apart from Δ, introduced in (4.12), our aim is to define the maps ς, ι, $\sigma_{1/2}$ and β in such a way that the whole diagram commutes and that the subsequence determined by the maps ς, $\sigma_{1/2}$ and Δ is exact.

Obviously, ς and ι are natural inclusions by the very definition of $H^0(X, \omega_X)$. For what concerns the map β, it suffices to define its image in one direct summand, i.e.

$$(\pi_h \circ \beta) : \bigoplus_{i<j} H^0(\omega_{C_{ij}}) \to H^0(\omega_{C_h}),$$

where π_h is the projection on the h^{th}-summand, for a given $h \in \{1, \dots, v\}$. When $h \neq i, j$, the image is 0, therefore the relevant summands are the following:

$$\begin{aligned} &\bigoplus_{h<j} H^0(\omega_{C_{hj}}) \oplus \bigoplus_{i<h} H^0(\omega_{C_{ih}}) \longrightarrow H^0(\omega_{C_h}), \\ &\left(\bigoplus_{h<j} \omega_{hj} \quad , \quad \bigoplus_{i<h} \omega_{ih} \right) \mapsto \sum_{h<j} \omega_{hj} - \sum_{i<h} \omega_{ih}, \end{aligned} \tag{4.23}$$

with the obvious condition that $\omega_{lm} = 0$ when $C_{lm} = \emptyset$. First of all observe that β is well-defined by the definition of $H^0(C_h, \omega_{C_h})$ (see Remark 4.3); moreover, the coefficients ± 1 are uniquely determined by the fact that $C_h \subset X_h$ and by Remark 4.10.

To define $\sigma_{1/2}$, recall that $\ker(f) \subseteq H^0(X, \omega_X) \subset \oplus_{i=1}^v H^0(\omega_{X_i}(C_i))$; thus, an element in $\ker(f)$ is a collection of v meromorphic 2-forms $(\gamma_1, \dots, \gamma_v) \in \oplus_{i=1}^v H^0(\omega_{X_i}(C_i))$ such that

$$\gamma_{ij} = -\gamma_{ji}, \text{ for } i < j,$$

and

$$\gamma^t_{\tau_k(i)\tau_k(j)\tau_k(k)} = 0 \text{ for each } k \neq i, \ j \text{ and for each } 1 \leqslant t \leqslant m_{\tau_k(i)\tau_k(j)\tau_k(k)},$$

where $\tau_k \in \mathrm{Sym}(\{i, j, k\})$ is the permutation which lexicographically reorders the index triple $\{i, j, k\}$. As before, we can limit ourselves to define its image on a given direct summand; therefore, if π_{ij} is the projection on the $(ij)^{\text{th}}$-summand, with $i < j$, then we have the following equivalent expressions

$$(\pi_{ij} \circ \sigma_{1/2})(\gamma_1, \dots, \gamma_v) = \frac{1}{2}(\mathcal{R}_{C_{ij}}(\gamma_i) - \mathcal{R}_{C_{ij}}(\gamma_j)) = \mathcal{R}_{C_{ij}}(\gamma_i) = -\mathcal{R}_{C_{ij}}(\gamma_j). \tag{4.24}$$

Observe that $\sigma_{1/2}$ is well-defined since the γ_i's are in the kernel of f; furthermore, when $C_{ij} = \emptyset$, the image is obviously 0.

By using the definition of Δ as in (4.13), it is straightforward to check that diagram (4.22) commutes. Furthermore, it is trivial to show that

$$\mathrm{Im}(\varsigma) = \ker(\sigma_{1/2}) \text{ and } \mathrm{Im}(\sigma_{1/2}) \subseteq \ker(\Delta).$$

To show the converse, take $\alpha \in \ker(\Delta)$, thus $(\oplus_i \delta_i(\beta(\alpha))) = 0$, i.e. $\beta(\alpha) \in \mathrm{Im}(\oplus_i(\mathcal{R}_{C_i}))$. This implies that $\alpha \in \mathrm{Im}(\sigma_{1/2})$.

From the fact that the subsequence in (4.22) is exact, it follows that

$$\ker(f) \cong \ker(\sigma_{1/2}) \oplus \mathrm{Im}(\sigma_{1/2}) \cong \mathrm{Im}(\varsigma) \oplus \ker(\Delta) \cong \bigoplus_{i=1}^d H^0(\omega_{X_i}) \oplus \ker(\Delta).$$

By Proposition 4.14, it follows that

$$\ker(f) \cong \bigoplus_{i=1}^d H^0(\omega_{X_i}) \oplus \mathrm{coker}(\Phi). \tag{4.25}$$

This proves (4.16).

To show that the map f is surjective, we have to reconstruct a global section of ω_X once we have a collection $\{\omega^t_{ijk}\} \in H_2(G_X, \mathbb{C})$.

Fix two indices $l < m$ in $\{1, \dots, d\}$, such that $C_{lm} \neq \emptyset$; this means that we consider the curve C_{lm} as the ambient variety to make our computations. From (4.20), we have three different possibilities:

- if $k < l$, then $\mathcal{R}_{P^t_{klm}}(\omega_{lm}) = \omega^t_{lmk} = \mathrm{sgn}((k, l, m))\omega^t_{klm} = \omega^t_{klm}$ for any $P^t_{klm} \in \Sigma_{klm}$, where (k, l, m) is a 3-cycle in $\mathrm{Sym}(\{k, l, m\})$;
- if $l < k < m$, then $\mathcal{R}_{P^t_{lkm}}(\omega_{lm}) = \omega^t_{lmk} = \mathrm{sgn}((m, k))\omega^t_{lkm} = -\omega^t_{lkm}$ for any $P^t_{lkm} \in \Sigma_{lkm}$, where (m, k) is a transposition in $\mathrm{Sym}(\{l, k, m\})$;

- if $k > m$, we directly have $\mathcal{R}_{P^t_{klm}}(\omega_{lm}) = \omega^t_{lmk}$ for any $P^t_{lmk} \in \Sigma_{lmk}$.

Therefore, by (4.21), on C_{lm} we have:

$$\sum_{k<l}\sum_{t=1}^{m_{klm}} \omega^t_{klm} - \sum_{k\in(l,m)\cap\mathbb{N}}\sum_{t=1}^{m_{lkm}} \omega^t_{lkm} + \sum_{k>m}\sum_{t=1}^{m_{lmk}} \omega^t_{lmk} = 0.$$

This means that the divisor

$$D = \sum_{k<l}\sum_{t=1}^{m_{klm}} \omega^t_{klm}P^t_{klm} - \sum_{k\in(l,m)\cap\mathbb{N}}\sum_{t=1}^{m_{lkm}} \omega^t_{lkm}P^t_{lkm} + \sum_{k>m}\sum_{t=1}^{m_{lmk}} \omega^t_{lmk}P^t_{lmk} \in Div(C_{lm}) \tag{4.26}$$

is homologous to zero. By the Residue Theorem, (4.26) implies there exists a global meromorphic 1-form $\omega_{lm} \in H^0(C_{lm}, \omega_{C_{lm}}(D))$ having the given residues at the points in $Supp(D)$, i.e. such that

$$\mathcal{R}_{P^r_{klm}}(\omega_{lm}) = \omega^r_{klm},\ \mathcal{R}_{P^s_{lkm}}(\omega_{lm}) = -\omega^s_{lkm},\ \mathcal{R}_{P^t_{lmk}}(\omega_{lm}) = \omega^t_{lmk},$$

where $1 \leqslant r \leqslant m_{klm}$, $1 \leqslant s \leqslant m_{lkm}$ and $1 \leqslant t \leqslant m_{lmk}$.

The above discussion obviously holds for each choice of index pairs. Fix now an index $1 \leqslant h \leqslant v$ and consider on the surface X_h the reducible nodal curve $C_h = X_h \cap \overline{(X \setminus X_h)}$; since we are on X_h, by Remark 4.10, we can write

$$C_h = C_h^+ + C_h^-,$$

where

$$C_h^- = \sum_{l<h} C_{lh},\ \text{and}\ C_h^+ = \sum_{m>h} C_{hm}.$$

Thus, by the above discussion, on each C_{lh} (resp. C_{hm}) we have a meromorphic 1-form $-\omega_{lh}$ (resp. ω_{hm}) inducing the given residues at the given triple points. Fix three indeces i, j, h and consider the set of triple points given by $X_i \cap X_j \cap X_h \neq \emptyset$. Since we are on the surface X_h, such set of points is determined by the intersection of $C_{ih} = C_{hi}$ and $C_{jh} = C_{hj}$. We have the following possibilities:

- if $i < j < h$, the set of triple points is Σ_{ijh} and on C_{ih} (resp., on C_{jh}) we have the meromorphic 1-form $-\omega_{ih}$ (resp., $-\omega_{jh}$); therefore, by (4.20),
$$\mathcal{R}_{P^t_{ijh}}(-\omega_{ih}) + \mathcal{R}_{P^t_{ijh}}(-\omega_{jh}) = \omega^t_{ijh} - \omega^t_{ijh} = 0,$$
for any $1 \leqslant t \leqslant m_{ijh}$;
- if $h < i < j$, the set of triple points is Σ_{hij} and on C_{hi} (resp., on C_{hj}) we have the meromorphic 1-form ω_{hi} (resp., ω_{hj}); as before,
$$\mathcal{R}_{P^t_{hij}}(\omega_{hi}) + \mathcal{R}_{P^t_{hij}}(\omega_{hj}) = \omega^t_{hij} - \omega^t_{hij} = 0,$$
for any $1 \leqslant t \leqslant m_{hij}$;
- if $i < h < j$, the set of triple points is Σ_{ihj} and on C_{ih} (resp., on C_{hj}) we have the meromorphic 1-form $-\omega_{ih}$ (resp., ω_{hj}); thus,
$$\mathcal{R}_{P^t_{ihj}}(-\omega_{ih}) + \mathcal{R}_{P^t_{ihj}}(\omega_{jh}) = -\omega_{ihj} + \omega_{ihj} = 0,$$
for any $1 \leqslant t \leqslant m_{ihj}$.

In either case, by Remark 4.3, such forms glue together to determine an element in $H^0(C_h, \omega_{C_h})$. This can be done for each $1 \leqslant h \leqslant d$, determining a collection $\{\overline{\omega}_h\} \in \bigoplus_{h=1}^v H^0(\omega_{C_h})$.

Assume now to be in the case of hypothesis (i), so each X_h is regular; since $h^1(X_h, \omega_{X_h}) = 0$, for each $1 \leqslant h \leqslant v$, by the exact sequences

$$0 \to \omega_{X_h} \to \omega_{X_h}(C_h) \to \omega_{C_h} \to 0,\ 1 \leqslant h \leqslant d,$$

the collection of forms $\{\overline{\omega}_h\}$ lifts up to a collection of forms $\{\omega_h\} \in \bigoplus_{h=1}^{v} H^0(\omega_{X_h}(C_h))$. Take an index pair with $h < k$; since C_{hk} is both a component of C_h^+ on X_h and of C_k^- on X_k, then

$$\mathcal{R}_{C_{hk}}(\omega_k) = -\mathcal{R}_{C_{hk}}(\omega_h).$$

This means that the collection $\{\omega_h\}$ is an element of $H^0(X, \omega_X)$, so the map f is surjective and formula (4.16) is proved.

Assume now to be in the case of hypothesis (ii); let X_j be an irreducible component of X, which is assumed to be an irregular surface. By Dolbeault cohomology, by the hypothesis on C_j and by the Kodaira vanishing theorem, we get the following diagram:

$$\begin{tikzcd} H^0(\omega_{X_j}(C_j)) \arrow[r, "\mathcal{R}_{C_j}"] & H^0(\omega_{C_j}) \arrow[r, "\delta_j"] & H^1(\omega_{X_j}) \arrow[r] & 0 , \\ & & H^0(X_j, \Omega^1_{X_j}) \arrow[ul, "\mathrm{tr}_{C_j}"] \arrow[u, "\eta_j"'] & \end{tikzcd} \tag{4.27}$$

which can be seen to be commutative, where tr_{C_j} is the trace map of holomorphic 1-forms on X_j to holomorphic 1-forms on C_j whereas η_j is the map defined by the cup product with the first Chern class of C_j, $\Upsilon_{C_j} \in H^{1,1}(X_j)$.

From the surjectivity of δ_j, it follows that

$$H^0(\omega_{C_j}) \cong \mathrm{Im}(\mathcal{R}_{C_j}) \oplus H^1(\omega_{X_j});$$

on the other hand, by the Hard Lefschetz theorem, η_j is an isomorphism. Since δ_j is injective on $\mathrm{Im}(\mathrm{tr}_{C_j})$, then $\mathrm{Im}(\mathcal{R}_{C_j}) \cap \mathrm{Im}(\mathrm{tr}_{C_j}) = \{0\}$. On the other hand,

$$\mathrm{Im}(\mathrm{tr}_{C_j}) \to H^1(\omega_{X_j})$$

is surjective. Then,

$$H^0(\omega_{C_j}) \cong \mathrm{Im}(\mathcal{R}_{C_j}) \oplus \mathrm{Im}(\mathrm{tr}_{C_j}).$$

Observe that the elements of $\mathrm{Im}(\mathrm{tr}_{C_j})$ give zero residues at the triple points of X lying on X_j, since such elements are restrictions to C_j of global holomorphic 1-forms of X_j. Therefore, the element $\omega_j \in H^0(\omega_{C_j})$ of the collection $\{\overline{\omega}_h\} \in \bigoplus_{h=1}^{v} H^0(\omega_{C_h})$, which was constructed from the given non-zero collection of residues in $H_2(G_X)$, is determined via $\mathcal{R}_{C_j}$ by an element in $H^0(\omega_{X_j}(C_j))$, which is necessarily not zero. Then we can conclude as above, proving also in this case the surjectivity of f. $\square$

In case X is a planar Zappatic surface, Theorem 4.15 implies the following:

Corollary 4.28. *Let $X = \bigcup_{i=1}^{v} \Pi_i$ be a planar Zappatic surface which has only E_3 points as Zappatic singularities. Then,*

$$p_g(X) = b_2(G_X), \tag{4.29}$$

$$q(X) = b_1(G_X). \tag{4.30}$$

Proof. Formula (4.29) trivially follows from Theorem 4.15. Notice that, in such a case, the proof of Theorem 3.15 becomes simpler. Indeed, each Σ_{ijk} (cf. Notation 3.3) is either a singleton or empty, since the double curves are lines (cf. Remark 4.7).

Formula (4.30) follows from (3.17), (4.29) and from the fact that $\chi(G_X) = 1 - b_1(G_X) + b_2(G_X)$. $\square$

In §5 we shall extend the above results to a good planar Zappatic surface X, assuming that X is smoothable, i.e. the central fibre of a degeneration.

5. Zappatic degenerations

In this section we will focus on degenerations of smooth surfaces to Zappatic ones.

Definition 5.1. Let Δ be the spectrum of a DVR (equiv. the complex unit disk). Then, a *degeneration* (of relative dimension n) is a proper and flat algebraic morphism

$$\begin{array}{c}\mathcal{X}\\ \downarrow{\scriptstyle\pi}\\ \Delta\end{array}$$

such that $\mathcal{X}_t = \pi^{-1}(t)$ is a smooth, irreducible, n-dimensional projective variety, for $t \neq 0$.

If Y is a smooth, projective variety, the degeneration

$$\begin{array}{lcl}\mathcal{X} & \subseteq & \Delta \times Y\\ \downarrow{\scriptstyle\pi} & & \\ \Delta & & \end{array}$$

is said to be an *embedded degeneration* in Y of relative dimension n. When it is clear from the context, we will omit the term embedded.

A degeneration (equiv. an embedded degeneration) is said to be *semistable* if the total space $\mathcal{X}$ is smooth and if the central fibre $\mathcal{X}_0$ (where 0 is the closed point of Δ) is a divisor in $\mathcal{X}$ with global normal crossings, i.e. $\mathcal{X}_0 = \sum V_i$ is a sum of smooth, irreducible components V_i's which meet transversally so that locally analitically the morphism π is defined by

$$(x_1, \dots, x_{n+1}) \xrightarrow{\pi} x_1 x_2 \cdots x_k = t \in \Delta, \quad k \leqslant n+1.$$

Given an arbitrary degeneration $\pi : \mathcal{X} \to \Delta$, the well-known Semistable Reduction Theorem (see [11]) states that there exists a base change $\beta : \Delta \to \Delta$ (defined by $\beta(t) = t^m$, for some m), a semistable degeneration $\psi : \mathcal{Z} \to \Delta$ and a diagram

$$\begin{tikzcd} \mathcal{Z} \arrow[r, dashed, "f"] \arrow[dr, "\psi"'] & \mathcal{X}_\beta \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ & \Delta \arrow[r, "\beta"] & \Delta \end{tikzcd}$$

such that f is a birational map obtained by blowing-up and blowing-down subvarieties of the central fibre. Therefore, statements about degenerations which are invariant under blowing-ups, blowing-downs and base-changes can be proved by directly considering the special case of semistable degenerations.

From now on, we will be concerned with degenerations of relative dimension two, namely degenerations of smooth, projective surfaces.

Definition 5.2. Let $\mathcal{X} \to \Delta$ be a degeneration (equiv. an embedded degeneration) of surfaces. Denote by $\mathcal{X}_t$ the general fibre, which is by definition a smooth, irreducible and projective surface; let $X = \mathcal{X}_0$ denote the central fibre. We will say that the degeneration is *Zappatic* if X is a Zappatic surface and $\mathcal{X}$ is smooth except for:

- ordinary double points at points of the double locus of X, which are not the Zappatic singularities of X;
- further singular points at the Zappatic singularities of X of type T_n, for $n \geqslant 3$, and Z_n, for $n \geqslant 4$.

A Zappatic degeneration will be called *good* if the central fibre is moreover a good Zappatic surface. Similarly, an embedded degeneration will be called a *planar Zappatic degeneration* if its central fibre is a planar Zappatic surface.

Notice that we require the total space $\mathcal{X}$ to be smooth at E_3-points of X.

If $\mathcal{X} \to \Delta$ is a good Zappatic degeneration, the singularities that $\mathcal{X}$ has at the Zappatic singularities of the central fibre X are explicitly described in [4].

Notation 5.3. Let $\mathcal{X} \to \Delta$ be a degeneration of surfaces and let $\mathcal{X}_t$ be the general fibre, which is by definition a smooth, irreducible and projective surface. Then, we consider some of the intrinsic invariants of $\mathcal{X}_t$:

- $\chi := \chi(\mathcal{O}_{\mathcal{X}_t})$;
- $K^2 := K^2_{\mathcal{X}_t}$;
- $p_g := p_g(\mathcal{X}_t)$;
- $\chi_{\text{top}} := \chi_{\text{top}}(\mathcal{X}_t)$;

If the degeneration is assumed to be embedded in $\mathbb{P}^r$, for some r, then we also have:

- $d := \deg(\mathcal{X}_t)$;
- $g := (K+H)H/2 + 1$, the sectional genus of $\mathcal{X}_t$.

We will be mainly interested in computing these invariants in terms of the central fibre X. For some of them, this is quite simple. For instance, when $\mathcal{X} \to \Delta$ is an embedded degeneration in $\mathbb{P}^r$, for some r, and if the central fibre $\mathcal{X}_0 = X = \bigcup_{i=1}^v X_i$, where the X_i's are smooth, irreducible surfaces of degree d_i, $1 \leqslant i \leqslant v$, then by the flatness of the family we have

$$d = \sum_{i=1}^v d_i.$$

When $\mathcal{X} \to \Delta$ is a good Zappatic degeneration (in particular a good, planar Zappatic degeneration), we can easily compute some of the above invariants by using our results of §3. Indeed, by using our Notation 3.10 and Propositions 3.12, 3.15, we get the following results.

Proposition 5.4. *Let $\mathcal{X} \to \Delta$ be a good Zappatic degeneration embedded in $\mathbb{P}^r$. Let $\mathcal{X}_0 = X = \bigcup_{i=1}^v X_i \subset \mathbb{P}^r$ be the central fibre and let $G = G_X$ be its associated graph. Let C be the double locus of X, i.e. the union of the double curves of X, $C_{ij} = C_{ji} = X_i \cap X_j$ and let $c_{ij} = \deg(C_{ij})$. Let D be a general hyperplane section of X and let D_i be the i^{th} smooth, irreducible component of D, which is a general hyperplane section of X_i, and denote by g_i its genus. Then:*

$$g = \sum_{i=1}^v g_i + \sum_{e_{ij} \in E} c_{ij} - v + 1. \tag{5.5}$$

When X is a good, planar Zappatic surface, if $G^{(1)}$ denotes the 1-skeleton of G, then:

$$g = 1 - \chi(G^{(1)}) = e - v + 1. \tag{5.6}$$

Proof. It directly follows from our computations in Proposition 3.12 and from the flatness of the family of hyperplane sectional curves of the degeneration (cf. formula (2.4)). ☐

Proposition 5.7. *Let $\mathcal{X} \to \Delta$ be a good Zappatic degeneration and let $\mathcal{X}_0 = X = \bigcup_{i=1}^v X_i$ be its central fibre. Let $G = G_X$ be its associated graph and let E the indexed set of edges of G. Let C be the double locus of X, which is the union of the double curves $C_{ij} = C_{ji} = X_i \cap X_j$. Denote by g_{ij} the genus of the smooth curve C_{ij}. Then*

$$\chi = \sum_{i=1}^v \chi(\mathcal{O}_{X_i}) - \sum_{e_{ij} \in E} \chi(\mathcal{O}_{C_{ij}}) + f. \tag{5.8}$$

Moreover, if $\mathcal{X} \to \Delta$ is a good, planar Zappatic degeneration, then

$$\chi = \chi(G) = v - e + f. \tag{5.9}$$

Proof. It follows from Proposition 3.15 and from the invariance of χ under flat degeneration. □

In the particular case that $\mathcal{X} \to \Delta$ is a semistable Zappatic degeneration, i.e. if X has only E_3-points as Zappatic singularities, then χ can be computed also in a different way by topological methods (see formula (5.13) in Theorem 5.12).

The above results are indeed more general: X is allowed to have any good Zappatic singularity, namely R_n-, S_n- and E_n-points, for any $n \geqslant 3$, and moreover our computations do not depend on the fact that X is smoothable, i.e. that X is the central fibre of a degeneration. Notice also that a good Zappatic degeneration is not semistable in general.

For what concerns the geometric genus, assume now — unless otherwise stated — that the Zappatic surface $X = \bigcup_{i=1}^{v} X_i$ is the central fibre of a semistable Zappatic degeneration $\mathcal{X} \to \Delta$, i.e. $\mathcal{X}$ is smooth and X has only E_3-points as Zappatic singularities. In this case, Theorem 4.15 implies the following:

Proposition 5.10. *Let $\mathcal{X} \to \Delta$ be a semistable Zappatic degeneration and $\mathcal{X}_0 = X = \bigcup_{i=1}^{v} X_i$ be its central fibre. Let G_X be the associated graph to X and Φ be the map defined in* (4.8). *Then, for any $t \in \Delta$,*

$$p_g(\mathcal{X}_t) \leqslant b_2(G_X) + \sum_{i=1}^{v} p_g(X_i) + \dim(\operatorname{coker}(\Phi)). \tag{5.11}$$

Proof. By semi-continuity, we have $p_g(\mathcal{X}_t) \leqslant p_g(\mathcal{X}_0) = p_g(X)$. One concludes by using formula (4.16). □

On the other hand, $p_g(\mathcal{X}_t)$ and $\chi(\mathcal{X}_t)$ can be also computed by topological methods: indeed the Clemens-Schmid exact sequence relates the mixed Hodge theory of the central fibre X to that of $\mathcal{X}_t$ by means of the monodromy of the total space $\mathcal{X}$ (see [15] for definitions and statements). In our particular situation, the following result holds:

Theorem 5.12 (Clemens-Schmid)**.** *Let $\mathcal{X} \to \Delta$ be a semistable Zappatic degeneration and $\mathcal{X}_0 = X = \bigcup_{i=1}^{v} X_i$ be its central fibre. Let G_X be the associated graph to X and Φ be the map defined in* (4.8). *Then, for any $t \neq 0$,*

$$\chi(\mathcal{O}_{\mathcal{X}_t}) = \chi(G_X), \tag{5.13}$$

$$p_g(\mathcal{X}_t) = b_2(G_X) + \sum_{i=1}^{d} p_g(X_i) + \dim(\operatorname{coker}(\Phi)). \tag{5.14}$$

A proof of Theorem 5.12 can be found in [15, "Clemens-Schmid I and II"]. The above result, together with Theorem 4.15, implies the following:

Corollary 5.15. *Let $X = \bigcup_{i=1}^{v} X_i$ be the central fibre of a semistable Zappatic degeneration $\mathcal{X} \to \Delta$. Then, for every $t \in \Delta$,*

$$p_g(\mathcal{X}_t) = p_g(X) = b_2(G_X) + \sum_{i=1}^{v} p_g(X_i) + \dim(\operatorname{coker}(\Phi)). \tag{5.16}$$

In particular, the geometric genus of the fibres of $\mathcal{X} \to \Delta$ is constant.

Proof. Formula 5.16 trivially follows from (4.16), (5.14) and from semicontinuity. □

Recalling the proof of Theorem 4.15, we also have the following:

Corollary 5.17. *Let $X = \bigcup_i X_i$ be a Zappatic surface with global normal crossings, i.e. with only E_3-points as Zappatic singularities. Let G_X be its associated graph. A necessary condition for the smoothability of X is the surjectivity of the homomorphism $f : H^0(X, \omega_X) \to H_2(G_X)$ defined in* (4.17).

Proof. If X is smoothable, then (5.16) implies that the equality in (4.16) holds. This implies that f is surjective by the proof of Theorem 4.15. ☐

The above results naturally suggest the following:

Question. Is the homomorphism $f : H^0(X, \omega_X) \to H_2(G_X)$ in (4.17) always surjective? Equivalently, does the equality in (4.16) always hold?

We believe that the answer to this question should be negative, but we have not been able to exhibit a counterexample so far.

In case the answer to the above question were negative, it should be interesting to compare the surjectivity of f with other smoothability conditions, like Friedman's one in [8].

We now generalize the computations for p_g to the case of good planar Zappatic degenerations, i.e. degenerations where the central fibre X is a union of planes having not only E_3-points, but also R_n-, S_n- and E_n-points for any $n \geqslant 3$.

Definition 5.18. Let $\mathcal{X} \to \Delta$ be a good planar Zappatic degeneration and $X = \mathcal{X}_0$ be its central fibre. Consider the semistable reduction $\mathcal{X}' \to \Delta$ of $\mathcal{X} \to \Delta$ together with its central fibre $X' = \mathcal{X}'_0$, which is a Zappatic surface with global normal crossings, i.e. with only E_3-points. We define the geometric genus as (cf. Remark 3.4):

$$p_g(X) := p_g(X') = h^0(X', \omega_{X'}). \tag{5.19}$$

As we will see in a moment, the definition is well-posed.

Theorem 5.20. *Let $X = \mathcal{X}_0$ be a good Zappatic surface which is the central fibre of a degeneration $\mathcal{X} \to \Delta$ and let G_X be its associated graph. Then*

$$p_g(X) = p_g(\mathcal{X}_t) = b_2(G_X) + \sum_{i=1}^{v} p_g(X_i) + \dim(\operatorname{coker}(\Phi)). \tag{5.21}$$

Sketch of the proof. Complete details will appear in [4]. Here we give an outline of the proof.

The first step is to understand how to get the semistable reduction locally near R_n-, S_n- and E_m-points, for $n \geqslant 3$ and $m \geqslant 4$.

Consider a R_n-point $x \in X$. Then x is an isolated singularity for the total space $\mathcal{X}$ and it is a *minimal singularity* in the sense of Kollár ([12] and [13]). Let $\tilde{\mathcal{X}} \to \mathcal{X}$ be the blow-up at x. Its exceptional divisor F is a minimal degree surface (of degree n) in $\mathbb{P}^{n+1} = \mathbb{P}(\mathcal{T}_{\mathcal{X},x})$, where $\mathcal{T}_{\mathcal{X},x}$ is the tangent space of $\mathcal{X}$ at x. Furthermore F is connected in codimension one and can be explicitly described. In particular one can show that either F is smooth or F has R_m-points, for $m \leqslant n$. Some points E_4 can appear along the intersection of the exceptional divisor with the strict transform of X. In any event, after finitely many blow-ups of the total space at points, we resolve the singularities of the total space. It turns out that all the components of the exceptional divisor are rational and all the double curves involved in them are also rational.

The situation is completely similar for S_n-points and E_n-points, $n \geqslant 4$.

Therefore one can get the semistable reduction $\mathcal{X}' \to \Delta$ of $\mathcal{X} \to \Delta$ just by blowing-up $\mathcal{X}$ at points which are good Zappatic singularities of the central fibre. Let $\sigma : \mathcal{X}' \to \mathcal{X}$ be this blow-up and $X' = \mathcal{X}'_0$ be the central fibre of $\mathcal{X}' \to \Delta$.

By (5.19), $p_g(X)$ is defined to be $p_g(X')$; Theorem 4.15 tells how to compute it.

Now, our second and last step is to prove that $p_g(X')$ is given by (5.21) and that $p_g(X') = p_g(\mathcal{X}_t)$, for $t \neq 0$. Since the semistable reduction of $\mathcal{X} \to \Delta$ involves only the central fibre and since all the exceptional divisors as well as all the curves involved in them are rational, it suffices to prove that $b_2(G_X) = b_2(G_{X'})$.

Consider an open n-face (resp. a n-angle) G_x of G_X, namely G_x is the subgraph with n vertices corresponding to n planes forming a R_n-point (resp. a S_n-point) x. Let G'_x be the subgraph of $G_{X'}$ containing the vertices corresponding to the proper transforms of the n planes and the exceptional divisors contained in $\sigma^{-1}(x)$. The above description of the infinitesimal neighbourhood of x shows that, as topological spaces, the subgraph G_x is a deformation retract of $G_{X'}$.

Similarly, if G_x is a closed n-face of G_X (i.e. a subgraph with n vertices corresponding to n planes forming an E_n-point x), then the above description shows that the subgraph G'_x of $G_{X'}$ containing the vertices corresponding to the proper transforms of the planes and the exceptional divisors contained in $\sigma^{-1}(x)$ is a triangulation of G_x.

It follows that the graphs G_X and $G_{X'}$ have the same homological invariants, which is what we had to prove. $\square$

References

[1] Artin, M., Winters, G., Degenerate fibres and stable reduction of curves, *Topology*, **10** (1971), 373–383.

[2] Bardelli F., Lectures on stable curves, in *Lectures on Riemann surfaces - Proc. on the college on Riemann surfaces, Trieste - 1987*, 648–704. Cornalba, Gomez-Mont, Verjovsky, World Scientific, Singapore, 1988.

[3] Barth W., Peters C., and Van de Ven A., *Compact Complex Surfaces*, Ergebnisse der Mathematik, 3. Folge, Band 4, Springer, Berlin, 1984.

[4] Calabri, A., Ciliberto, C., Flamini, F., Miranda, R., On degenerations of projective surfaces to unions of planes, in progress.

[5] Ciliberto, C., Lopez, A.F., Miranda, R., Projective degenerations of $K3$ surfaces, Gaussian maps and Fano threefolds, *Invent. Math.*, **114** (1993), 641–667.

[6] Ciliberto, C., Miranda, R., Teicher, M., Pillow degenerations of $K3$ surfaces, in Ciliberto et al. (eds.), *Application of Algebraic Geometry to Computation, Physics and Coding Theory*, Nato Science Series II/36, Kluwer Academic Publishers, 2002.

[7] Ciliberto, C., Miranda, R., Teicher, M., Pillow degenerations of $K3$ surfaces, preprint math.AG, n. 020929.

[8] Friedman, R., Global smoothings of varieties with normal crossings, *Ann. Math.*, **118** (1983), 75–114.

[9] Friedman, R., Morrison, D.R., (eds.,) *The birational geometry of degenerations*, Progress in Mathematics 29, Birkhauser, Boston, 1982.

[10] Griffiths, P., Harris, J., *Principles of Algebraic Geometry*, Wiley Classics Library, New York, 1978.

[11] Kempf, G., Knudsen, F.F., Mumford, D., and Saint-Donat, B., Toroidal embeddings. I., Lecture Notes in Mathematics 339, Springer-Verlag, Berlin-New York, 1973.

[12] Kollár, J., Toward moduli of singular varieties, *Compositio Mathematica*, **56** (1985), 369–398.

[13] Kollár, J., Shepherd-Barron, N.I., Threefolds and deformations of surface singularities, *Invent. math.*, **91** (1988), 299–338.

[14] Hartshorne, R., *Families of curves in $\mathbb{P}^3$ and Zeuthen's problem*, Mem. Amer. Math. Soc. **130** (1997), no. 617.

[15] Morrison, D.R., The Clemens-Schmid exact sequence and applications, in *Topics in Trascendental Algebraic Geometry, Ann. of Math. Studies*, **106** (1984), 101–119.

[16] Persson, U., *On degeneration of algebraic surfaces*, Memoirs of the American Mathematical Society, **189**, AMS, Providence, 1977.

[17] Severi, F., *Vorlesungen über algebraische Geometrie*, vol. 1, Teubner, Leipzig, 1921.

[18] Zappa, G., Caratterizzazione delle curve di diramazione delle rigate e spezzamento di queste in sistemi di piani, *Rend. Sem. Mat. Univ. Padova*, **13** (1942), 41–56.

[19] Zappa, G., Su alcuni contributi alla conoscenza della struttura topologica delle superficie algebriche, dati dal metodo dello spezzamento in sistemi di piani, *Acta Pont. Accad. Sci.*, **7** (1943), 4–8.

[20] Zappa, G., Applicazione della teoria delle matrici di Veblen e di Poincaré allo studio delle superficie spezzate in sistemi di piani, *Acta Pont. Accad. Sci.*, **7** (1943), 21–25.

[21] Zappa, G., Sulla degenerazione delle superficie algebriche in sistemi di piani distinti, con applicazioni allo studio delle rigate, *Atti R. Accad. d'Italia, Mem. Cl. Sci. FF., MM. e NN.*, **13** (2) (1943), 989–1021.

[22] Zappa, G., Invarianti numerici d'una superficie algebrica e deduzione della formula di Picard-Alexander col metodo dello spezzamento in piani, *Rend. di Mat. Roma*, **5** (5) (1946), 121–130.

[23] Zappa, G., Alla ricerca di nuovi significati topologici dei generi geometrico ed aritmetico di una superficie algebrica, *Ann. Mat. Pura Appl.*, **30** (4) (1949), 123–146.

[24] Zappa, G., Sopra una probabile disuguaglianza tra i caratteri invariantivi di una superficie algebrica, *Rend. Mat. e Appl.*, **14** (1955), 1–10.

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