THE RANK OF THE 2ND GAUSSIAN MAP FOR GENERAL CURVES

ALBERTO CALABRI, CIRO CILIBERTO AND RICK MIRANDA

ABSTRACT. We prove that, for the general curve of genus g, the 2nd Gaussian map μ is injective if $g \leqslant 17$ and surjective if $g \geqslant 18$. The proof relies on the study of the limit of μ when the general curve of genus g degenerates to a general stable binary curve, i.e. the union of two rational curves meeting at $g+1$ points.

INTRODUCTION

Let X be a smooth, projective curve of genus g and let $\mathcal{L}$ be a line bundle on X. Consider the product $X \times X$, with the projections p_1, p_2 to the factors, and the natural morphism p to the symmetric product $X(2)$. One has $p_*(p_1^*\mathcal{L} \otimes p_2^*\mathcal{L}) = \mathcal{L}^+ \oplus \mathcal{L}^-$, where $\mathcal{L}^\pm$ are the invariant and anti-invariant line bundles with respect to the involution $(x,y) \mapsto (y,x)$. One has $H^0(\mathcal{L}^+) \cong \mathrm{Sym}^2 H^0(\mathcal{L})$ and $H^0(\mathcal{L}^-) \cong \wedge^2 H^0(\mathcal{L})$. Restriction to the diagonal of $X(2)$ gives rise to two maps

$$\mu_{\mathcal{L},1} : \mathrm{Sym}^2 H^0(\mathcal{L}) \to H^0(\mathcal{L}^{\otimes 2}), \quad w_{\mathcal{L},1} : \wedge^2 H^0(\mathcal{L}) \to H^0(\mathcal{L}^{\otimes 2} \otimes K_X),$$

where K_X is the canonical bundle of X. Both maps have a well known geometric meaning. The former is given by considering the map $\phi_{\mathcal{L}} : X \to \mathbb{P}^r := \mathbb{P}(H^0(\mathcal{L}))^*$ defined by the complete linear series determined by $\mathcal{L}$ and by pulling back to X forms of degree two in $\mathbb{P}^r$. The latter is given by considering the composition γ of $\phi_{\mathcal{L}}$ with the *Gauss map* of X to the Grassmannian of lines $\mathbb{G}(1,r)$ and by pulling back to X via γ forms of degree one in $\mathbb{P}^{\binom{r+1}{2}-1}$.

The maps $\mu_{\mathcal{L},1}$ and $w_{\mathcal{L},1}$ are the first instances of two hierarchies of maps $\mu_{\mathcal{L},k}$ and $w_{\mathcal{L},k}$, defined for all positive integers k, and called by some authors *higher Gaussian maps* of X. They are inductively defined by iterated restrictions to the diagonal of $X(2)$. Precisely for all $k \geqslant 2$ one has

$$\mu_{\mathcal{L},k} : \ker(\mu_{\mathcal{L},k-1}) \to H^0(\mathcal{L}^{\otimes 2} \otimes K_X^{\otimes 2(k-1)}), \quad w_{\mathcal{L},k} : \ker(w_{\mathcal{L},k-1}) \to H^0(\mathcal{L}^{\otimes 2} \otimes K_X^{\otimes (2k-1)}).$$

These maps are particularly interesting when $\mathcal{L} \cong K_X$, in which case we will simply denote them as μ_k and w_k. They are both defined at a general point of the moduli space of curves $\mathcal{M}_g$ and it is natural to guess that they have some modular meaning. Indeed, μ_1 is the codifferential, at the point corresponding to X, of the Torelli map $\tau : \mathcal{M}_g \to \mathcal{A}_g$, and Noether's theorem says it is surjective if and only if X is non-hyperelliptic.

The map w_1 is called the *Wahl map*, and it is related to important deformation and extendability properties of the canonical image of the curve (cf. [BM, W]). Because of this, it has been studied by various authors, too many to be quoted here. One the most interesting results concerning it is perhaps a theorem first proved by Ciliberto, Harris and Miranda in [CHM], to the effect that w_1 is surjective, as expected, for a general curve of genus $g = 10$ and $g \geqslant 12$. Moreover, this map is injective, as expected, for a general curve of genus $g \leqslant 8$, cf. [CM2]. Unexpectedly, the Wahl map is not of maximal rank for a general curve of genus $g = 9, 11$.

In general, all maps μ_k and w_k are supposed to be meaningful in the geometry of curves, especially of curves with general moduli. Here we will look in particular at the map $\mu_2 \colon \mathcal{I}_2(K_X) \to H^0(X, K_X^{\otimes 4})$, where $\mathcal{I}_2(K_X)$ is the vector space of forms of degree two vanishing on the canonical model of X. From now on we will simply denote this map by μ, and we will call it the *2nd Gaussian map* of X. This map was first considered by Green-Griffiths in [G] and its importance resides in the fact that it is related to the 2nd fundamental form of the moduli space of curves $\mathcal{M}_g$, embedded in $\mathcal{A}_g$ via the Torelli map, cf. [CPT, CF1, CF2].

Despite the unexpected behaviour of the Wahl map for genus $g = 9, 11$, a reasonable working hypothesis is that the 2nd Gaussian map μ should be of maximal rank for a general curve of any genus g. A dimension

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count shows that this is equivalent to say that μ should be injective for a general curve of genus $g \leqslant 17$ and surjective if $g \geqslant 18$. So far, the best result in this direction has been proved by Colombo and Frediani in [CF3], where, by studying hyperplane sections of high genera of K3 surfaces, they show that μ is surjective for a general curve of genus $g > 152$. For other interesting results concerning μ, see also [CF2, CFP].

In this paper, we prove the maximal rank property for every genus:

Theorem 1. *The 2nd Gaussian map $\mu\colon \mathcal{I}_2(K_X) \to H^0(X, K_X^{\otimes 4})$ for X a general curve of any genus g has maximal rank, namely it is injective for $g \leqslant 17$ and surjective for $g \geqslant 18$.*

As shown in [CPT], the map μ has a lifting $\rho : \mathcal{I}_2(K_X) \to \mathrm{Sym}^2(H^0(K_X^{\otimes 2}))$, which is the datum of the second fundamental form of the Torelli embedding at the point corresponding to X in the non-hyperelliptic case. As proved in [CF2], Corollary 3.4, ρ is injective for *all* non-hyperelliptic curves X. Our result shows that if X is general, then the image of ρ is transversal to the kernel of the multiplication map $\mathrm{Sym}^2(H^0(K_X^{\otimes 2})) \to H^0(K_X^{\otimes 4})$.

The proof of Theorem 1 is by degeneration to a reducible nodal curve for which the limit of μ, described in §1, has maximal rank. The theorem then follows by upper semicontinuity. We do not use graph curves here, i.e. the curves exploited in [CHM], because for them the limit of μ is more difficult to understand. We used instead a general *binary curve*, i.e. a stable curve of genus g consisting of two rational components meeting at $g+1$ points, which are general on both components. For such a curve C we explicitly write down the ideal $\mathcal{I}_2(K_C)$ in §2. In §3 we describe the 2nd Gaussian map for C modulo torsion, and then, in §4, we deal with the torsion part. By direct computations performed with Maple (the script is presented and commented in the Appendix), we verified the injectivity for a general binary curve of genus $g \leqslant 17$ and the surjectivity for $g = 18$. Finally, in §5, we proceed by induction on g to complete the argument for $g \geqslant 19$.

The behaviour of μ, and its connection with the curvature of $\mathcal{M}_g$ in $\mathcal{A}_g$, indicates possible relations of the surjectivity of μ with the Kodaira dimension of $\mathcal{M}_g$ being non-negative. This, we think, would be a great subject for future research. Also interesting is the study of the Gaussian maps μ_k, w_k for higher values of k. The maps μ_k are related to higher fundamental forms of the Torelli immersion of $\mathcal{M}_g$ in $\mathcal{A}_g$ at a non-hyperelliptic point. Are these maps also of maximal rank for a general curve?

In this paper we work over the complex field and we will use standard notation in algebraic geometry. In particular, if X is a Gorenstein curve, Ω^1_X will denote its sheaf of Kähler differentials and K_X will denote its dualizing sheaf or canonical bundle, or a canonical divisor. In general, we will indifferently use sheaf, bundle or divisor notation. We will often write $H^i(\mathcal{L})$ instead of $H^i(X, \mathcal{L})$ for cohomology spaces.

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1. The 2nd Gaussian map for a stable curve

Let X be a stable curve of genus g. We will denote by $\mathcal{I}_2(K_X)$ the vector space of forms of degree 2 vanishing on the canonical model of X. If X is smooth, the 2nd Gaussian map $\mu\colon \mathcal{I}_2(K_X) \to H^0(X, K_X^{\otimes 4})$ is locally defined as follows.

Fix a basis $\{\omega_i\}$ of $H^0(K_X)$, and write it in a local coordinate z as $\omega_i = f_i(z)\,dz$. Let $Q \in \mathcal{I}_2(K_X)$, with $Q = \sum_{i,j} s_{ij}\omega_i \otimes \omega_j$, the matrix (s_{ij}) being symmetric. Since $\sum_{i,j} s_{ij} f_i f_j \equiv 0$, one has $\sum_{i,j} s_{ij} f_i' f_j \equiv 0$. The local expression of $\mu(Q)$ is then (cf., e.g., [CF2])

$$\mu(Q) = \sum_{i,j} s_{ij} f_i'' f_j\,(dz)^4 = -\sum_{i,j} s_{ij} f_i' f_j'\,(dz)^4. \tag{1}$$

If X is nodal, one can similarly define the 2nd Gaussian map

$$\mu\colon \mathcal{I}_2(K_X) \to H^0(X, \mathrm{Sym}^2(\Omega^1_X) \otimes K_X^{\otimes 2})$$

which is locally defined in a similar way as in (1). Precisely, let $\{\omega_i\}$ be a basis of $H^0(K_X)$. In local coordinates, we can write $\omega_i = f_i\,\xi$, where f_i is a regular function and ξ is a local generator of the canonical bundle K_X. Then μ is locally defined by

$$\mu(Q) = -\sum_{i,j} s_{ij}\,df_i\,df_j\,\xi^{\otimes 2}. \tag{2}$$

Given a flat degeneration over a disc of a general curve to a stable curve X, the 2nd Gaussian map for X is the flat limit of the 2nd Gaussian map for the general curve.

It is useful to describe in some detail the space $H^0(X, \mathrm{Sym}^2(\Omega^1_X)\otimes K_X^{\otimes 2})$. First remark that $\mathrm{Sym}^2(\Omega^1_X)$ has torsion T supported at the nodes of X. So we have a short exact sequence

$$0 \to T \to \mathrm{Sym}^2(\Omega^1_X) \to \mathcal{F}_X \to 0,$$

where $\mathcal{F}_X$ is a non-locally free, rank 1, torsion free sheaf on X.

Lemma 2. *(a) For every node p of X, T_p is a 3-dimensional vector space; if the local equation of X around p is $xy = 0$, then T_p is spanned by $dx\,dy$, $x\,dx\,dy$ and $y\,dx\,dy$.*
(b) If X_i are the irreducible components of the normalization $\pi\colon \tilde{X} \to X$ of X, one has

$$\mathcal{F}_X \cong \bigoplus\nolimits_i \pi_* K_{X_i}^{\otimes 2}.$$

Proof. Since $y\,dx = -x\,dy$, a local section of $\mathrm{Sym}^2(\Omega^1_X)$ around a node $xy = 0$ can be uniquely written as $f(x)\,(dx)^2 + g(x,y)\,dx\,dy + h(y)\,(dy)^2$, where $g(x,y)$ is linear. Then (a) is a local computation and (b) follows from (a). □

As a consequence, since $K_{X|X_i} = K_{X_i}(D_i)$ where D_i be the divisor of nodes on X_i, one has

$$H^0(X, \mathrm{Sym}^2(\Omega^1_X) \otimes K_X^{\otimes 2}) \cong T \oplus \bigoplus\nolimits_i H^0(X_i, K_{X_i}^{\otimes 4}(2D_i)). \tag{3}$$

where $T \cong \mathbb{C}^{3\delta}$, with δ the number of nodes of X.

2. Canonical binary curves

Let $[x_1, \dots, x_g]$ be homogenous coordinates in $\mathbb{P}^{g-1}$, $g \geqslant 3$. Let $p_h = [0, \dots, 0, 1, 0, \dots, 0]$, with 1 at the h-th place, $1 \leqslant h \leqslant g$, be the coordinate points and $u = [1, 1, \dots, 1]$ the unit point. Take C_1, C_2 two distinct rational normal curves in $\mathbb{P}^{g-1}$ passing through p_h, $1 \leqslant h \leqslant g$, and u. Then C_1, C_2 intersect transversally at these $g+1$ points and have no further intersection.

We may and will assume that C_k, $k = 1, 2$, is the closure of the image of the map f_k given by

$$t \mapsto f_k(t) = \left[\frac{1}{t-\alpha_{k,1}}, \frac{1}{t-\alpha_{k,2}}, \dots, \frac{1}{t-\alpha_{k,g}}\right], \tag{4}$$

where $\alpha_{k,i} \in \mathbb{C}$, $k = 1, 2$, $i = 1, \dots, g$. In particular, $f_k(\alpha_{k,h}) = p_h$, $h = 1, \dots, g$, and $f_k(\infty) = u$. For our purposes, the $\alpha_{k,i}$'s will be general in $\mathbb{C}$. Actually, we will often consider them as indeterminates on $\mathbb{C}$.

The curve $C = C_1 \cup C_2$ is the limit of a general canonical curve $X \subset \mathbb{P}^{g-1}$ of genus g, and C is canonical too, i.e. $\mathcal{O}_C(1) \cong K_C$. The curve C is usually called a *canonical binary curve*.

Proposition 3. *A canonical binary curve $C = C_1 \cup C_2$ is projectively normal.*

Proof. The assertion is trivial for $g = 3$, which is the minimum allowed value of g. So we may assume $g \geq 4$. By Theorem 1.2 in [S], it suffices to show that there are $g-2$ smooth points of C spanning a $\mathbb{P}^{g-3}$ which meets C scheme-theoretically at these $g-2$ points only. Choose $g-2$ general points on C_1 and let $\Lambda \cong \mathbb{P}^{g-3}$ be their span. This meets transversally C_1 at these points. We claim that Λ does not meet C_2. Otherwise choose $g-4$ general points on C_1 and project C down to $\mathbb{P}^3$ from their span. The image of C_1 is a rational normal cubic Γ_1, whereas C_2 projects birationally (cf. [CC]) to a non-degenerate rational curve Γ_2 of degree larger than 3, thus Γ_1 and Γ_2 are distinct. Moreover the general secant line to Γ_1 would meet Γ_2, which is impossible by the trisecant lemma (see the *focal proof* in [ChC]). □

Remark 4. The only information that we will need from the above proposition is that C is quadratically normal, which is equivalent to

$$\dim(\mathcal{I}_2(K_C)) = \binom{g-2}{2}.$$

The simple argument in the proof of Proposition 3 relies on Schreyer's result, which requires a careful analysis, following the classical approach of Petri. The same result would follow by proving that the general hyperplane section of C verifies the general position theorem (see [ACGH], p. 109). This may be proved with the same argument as above, but we do not dwell on this here.

In case C is a general binary curve, it is quite simple to prove that C is quadratically normal. One way is to remark that the general trigonal binary curve is quadratically normal. For example, if $g = 2h$, embed $\mathbb{F}_0$ in $\mathbb{P}^{g-1}$ via the linear system of curves of type $(1, h-1)$. The general trigonal binary curve is the union of the images of a general curve of type $(1, h)$ and of a general curve of type $(2, 1)$. The case g odd is similar and is left to the reader.

We are now interested in explicitly describing the vector space $\mathcal{I}_2(K_C)$ of degree two forms vanishing on C, i.e. the domain of the map μ for C. The analysis we are going to make will provide another proof that the general binary curve C is quadratically normal.

For $k = 1, 2$, set

$$A_k(t) = \prod_{i=1}^{g}(t - \alpha_{k,i}). \tag{5}$$

For each $h = 0, \dots, g$, the coefficients $c_{k,h}$ of t^{g-h} in $A_k(t)$ are, up to sign, the elementary symmetric functions

$$c_{k,0} = 1, \qquad c_{k,h} = (-1)^h \sum_{1 \leqslant i_1 < i_2 < \cdots < i_h \leqslant g} \alpha_{k,i_1}\alpha_{k,i_2} \cdots \alpha_{k,i_h}.$$

Note that the index h is the degree of $c_{k,h}$ as a polynomial in the $\alpha_{k,i}$'s .

Fix $k \in \{1, 2\}$. Since C_k passes through the coordinate points, the equation of a quadric $Q \subset \mathbb{P}^{g-1}$ containing C_k has the form

$$\sum_{1 \leqslant i < j \leqslant g} s_{ij} x_i x_j = 0, \tag{6}$$

with the conditions

$$P_k(t) := \sum_{1 \leqslant i < j \leqslant g} \frac{A_k(t)}{(t - \alpha_{k,i})(t - \alpha_{k,j})}\, s_{ij} = \sum_{n=0}^{g-2} P_{k,n} t^n \equiv 0,$$

where $P_k(t)$ is a polynomial in t of degree $g - 2$ whose coefficients are linear polynomials $P_{k,n}(s_{ij})$ in the s_{ij}'s, $n = 0, \dots, g - 2$. By expanding the product $A_k(t)$ one sees that the coefficients $p_{k,h;i,j}$ of s_{ij} in $P_{k,g-2-h}$, $h = 0, \dots, g - 2$, are

$$p_{k,0;i,j} = 1, \quad p_{k,1;i,j} = -\sum_{i_1 \neq i,j} \alpha_{k,i_1}, \quad p_{k,h;i,j} = (-1)^h \sum_{\substack{i_1 < i_2 < \cdots < i_h \\ \text{all} \neq i,j}} \alpha_{k,i_1}\alpha_{k,i_2} \cdots \alpha_{k,i_h},\ 2 \leqslant h \leqslant g - 2, \tag{7}$$

namely the elementary symmetric functions, removing the i and j terms, up to sign. Again the index h coincides with the degree of $p_{k,h;i,j}$ as a homogeneous polynomial in the $\alpha_{k,i}$'s.

Consider also the polynomials

$$Q_{k,n}(s_{ij}) := \sum_{1 \leqslant i < j \leqslant g} \left(\sum_{m=0}^{g-2-n} \alpha_{k,i}^m \alpha_{k,j}^{g-2-n-m} \right) s_{ij}, \qquad n = 0, \dots, g - 2,$$

and let $q_{k,h;i,j} = \sum_{m=0}^{h} \alpha_{k,i}^m \alpha_{k,j}^{h-m}$ be the coefficient of s_{ij} in $Q_{k,g-2-h}$, $h = 0, \dots, g - 2$. Also in this case the index h coincides with the degree of $q_{k,h;i,j}$ as a homogeneous polynomial in the $\alpha_{k,i}$'s.

Remark 5. The coefficient $q_{k,h;i,j}$ of s_{ij} in $Q_{k,g-2-h}$ can be recursively computed by

$$q_{k,0;i,j} = 1, \quad q_{k,1;i,j} = \alpha_{k,i} + \alpha_{k,j}, \quad q_{k,h;i,j} = q_{k,1;i,j} q_{k,h-1;i,j} - \alpha_{k,i}\alpha_{k,j} q_{k,h-2;i,j}, \quad 2 \leqslant h \leqslant g - 2.$$

Note that all the monomials $\alpha_{k,j}^m \alpha_{k,i}^{h-m}$, $m = 0, \dots, h$, in particular $\alpha_{k,i}^h$ and $\alpha_{k,j}\alpha_{k,i}^{h-1}$, appear in $q_{k,h;i,j}$ with coefficient 1. Note also the recursive formula

$$q_{k,h;i,j} = \alpha_i q_{h-1;i,j} + \alpha_j^h, \quad 1 \leqslant h \leqslant g - 2. \tag{8}$$

We will need the following lemma:

Lemma 6. *Fix $k \in \{1, 2\}$. For each $n = 0, \dots, g - 2$, one has*

$$P_{k,n} = \sum_{m=0}^{g-2-n} c_{k,m} Q_{k,n+m}. \tag{9}$$

In particular, the linear system

$$P_{k,n}(s_{ij}) = 0, \qquad n = 0, \dots, g - 2, \tag{10}$$

in the s_{ij}'s is equivalent to the linear system

$$Q_{k,n}(s_{ij}) = 0, \qquad n = 0, \dots, g - 2. \tag{11}$$

Proof. One has $P_{k,g-2} = Q_{k,g-2}$ and $P_{k,g-3} = Q_{k,g-3} + c_{k,1}Q_{k,g-2}$. Next we proceed by induction: formula (9) is equivalent to

$$p_{k,h;i,j} = \sum_{l=0}^{h} c_{k,l} q_{k,h-l;i,j}, \qquad \text{for} \quad h = 0, \dots, g-2. \tag{12}$$

For $h = 0, 1$, (12) clearly holds. Since the index k is fixed, we omit it. For $2 \leqslant h \leqslant g-2$, one has

$$\begin{aligned} p_{h;i,j} - c_h q_{0;i,j} &= (\alpha_i + \alpha_j) p_{h-1;i,j} - \alpha_i \alpha_j p_{h-2;i,j} \overset{\text{(by induction)}}{=} \\ &= c_{h-1} q_{1;i,j} + \sum_{l=0}^{h-2} c_l (q_{h-l-1;i,j} q_{1;i,j} - \alpha_i \alpha_j q_{h-l-2;i,j}) = \sum_{l=0}^{h-1} c_l q_{h-l;i,j}, \end{aligned}$$

which proves (12) and therefore (9). Since $c_{k,0} = 1$, the base change matrix between the $Q_{k,n}$'s and the $P_{k,n}$'s is unipotent triangular, hence it is invertible. The equivalence between (10) and (11) follows. □

Next we can give the announced description of $\mathcal{I}_2(K_C)$.

Proposition 7. *Let $g \geqslant 3$. For a general choice of $\alpha_{k,i}$, $1 \leqslant k \leqslant 2$, $1 \leqslant i \leqslant g$, one has that*

(a) the linear system (11) *has maximal rank $g-1$;*

(b) the linear system

$$Q_{1,0}(s_{ij}) = \cdots = Q_{1,g-2}(s_{ij}) = Q_{2,0}(s_{ij}) = \cdots = Q_{2,g-3}(s_{ij}) = 0, \tag{13}$$

has maximal rank $2g-3$.

Proof. (a) Since the index k is fixed, we drop it here. Let us consider the matrix

$$U := U(\alpha_1, \dots, \alpha_g) = (q_{h;i,j})_{0 \leqslant h \leqslant g-2, 1 \leqslant i < j \leqslant g}$$

of size $(g-1) \times \binom{g}{2}$, where the pairs (i,j) are lexicographically ordered. We have to prove that there is a minor of U of order $g-1$ which is not identically zero. We show this for the minor $D := D(\alpha_1, \dots, \alpha_g)$ determined by the first $g-1$ columns, indexed by $(1,i)$ with $2 \leqslant i \leqslant g$. This is true if $g = 3$, so we proceed by induction on g. Look at D as a polynomial in α_g: it has degree $g-2$ and the coefficient of α_g^{g-2} is $D(\alpha_1, \dots, \alpha_{g-1})$ (cf. Remark 5), which is non-zero by induction. This proves the assertion.

Equivalently, by subtracting from each row the previous one multiplied by α_1 and using (8) (cf. Remark 5), one sees that D is the Vandermonde determinant $V(\alpha_2, \dots, \alpha_g) = \prod_{2 \leqslant i < j \leqslant g} (\alpha_j - \alpha_i)$ of $\alpha_2, \dots, \alpha_g$.

(b) We use the same idea of the proof of (a). Form a matrix $Z := Z(\alpha_{k,i})_{1 \leqslant k \leqslant 2, 1 \leqslant i < j \leqslant g}$ of size $(2g-3) \times \binom{g}{2}$ by concatenating vertically U (for $k = 1$) and the matrix

$$W := W(\alpha_{2,1}, \dots, \alpha_{2,g}) = (q_{2,h;i,j})_{1 \leqslant h \leqslant g-2, 1 \leqslant i < j \leqslant g}.$$

It suffices to prove that the minor $M := M(\alpha_{k,i})_{1 \leqslant k \leqslant 2, 1 \leqslant i < j \leqslant g}$ of Z determined by the first $2g-3$ columns, indexed by $(1,i), (2,j)$ with $2 \leqslant i \leqslant g$ and $3 \leqslant j \leqslant g$, is not identically zero as a polynomial in the $\alpha_{k,i}$'s. This is clearly true for $g = 3$, so we proceed by induction on g. Look at M as a polynomial in $\alpha_{1,g}$ and $\alpha_{2,g}$: one sees that M has degree $g-2$ in both $\alpha_{1,g}$, $\alpha_{2,g}$ and the monomials of the highest degree are $\alpha_{k,g}^{g-2} \alpha_{3-k,g}^{g-3}$, $1 \leqslant k \leqslant 2$, whose coefficient is $(-1)^k (\alpha_{3-k,1} - \alpha_{3-k,2}) M(\alpha_{k,i})_{1 \leqslant k \leqslant 2, 1 \leqslant i < j \leqslant g-1}$, which is non-zero by induction, proving the assertion.

Equivalently, looking at M as a polynomial in $\alpha_{1,1}$, one sees that the coefficient of the highest degree monomial $\alpha_{1,1}^{g-2}$ is the product of the two Vandermonde determinants $V(\alpha_{2,2}, \dots, \alpha_{2,g}) V(\alpha_{1,3}, \dots, \alpha_{1,g})$. □

Remark 8. The solutions of the linear system (11), as well as those of (10), give us the quadrics containing the rational normal curve C_k, whereas the solutions of (13) give us the quadrics in $\mathcal{I}_2(K_C)$ for the binary curve $C = C_1 \cup C_2$.

3. Binary curves: the 2nd Gaussian map modulo torsion

Let $C = C_1 \cup C_2$ be a general binary curve. In this section we will consider the composition ν of the 2nd Gaussian map for C with the projection to the non-torsion part of $H^0(C, \mathrm{Sym}^2(\Omega^1_C) \otimes K_C^{\otimes 2})$ (cf. formula (3) in §1). Specifically, for $k = 1, 2$, we will look at the map

$$\nu_k \colon \mathcal{I}_2(K_C) \to H^0(C_k, K_{C_k}^{\otimes 4}(2D_k))$$

where D_k is a divisor of degree $g+1$ on C_k, therefore

$$H^0(C_k, K_{C_k}^{\otimes 4}(2D_k)) \cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2g-6)),$$

and $\nu = \nu_1 \oplus \nu_2$.

The map ν_k can be explicitly written down, by taking into account (2) and the description of the ideal $\mathcal{I}_2(K_C)$ (see §2). Precisely, the let $Q \in \mathcal{I}_2(K_C)$ be of the form (6) where the s_{ij}'s are solutions of (13). Then

$$\nu_k(Q) = \sum_{1\leqslant i<j\leqslant g} \frac{1}{(t-\alpha_{k,i})^2(t-\alpha_{k,j})^2}\, s_{ij}(dt)^4 \in H^0(C_k, K_{C_k}^{\otimes 4}(2D_k)).$$

To look at this as a section of $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2g-6))$, we multiply by $A_k^2(t)$. Hence

$$\nu_k(Q) = \sum_{1\leqslant i<j\leqslant g} \frac{A_k^2(t)}{(t-\alpha_{k,i})^2(t-\alpha_{k,j})^2}\, s_{ij} =: R_k(t) \tag{14}$$

is a polynomial in t whose apparent degree is $2g-4$, but its coefficient of degree $2g-4$ is $P_{k,g-2}$ and the one of degree $2g-5$ is proportional to $P_{k,g-3}$, hence they vanish and $R_k(t)$ has actual degree $2g-6$.

Using this explicit description (14) of ν, we asked Maple to compute its rank for low values of g (see the Appendix for Maple scripts). The result is the following:

Proposition 9. *The map ν has maximal rank for $g \leqslant 18$, namely ν is injective for $g \leqslant 10$ and it is surjective for $11 \leqslant g \leqslant 18$.* □

Corollary 10. *The 2nd Gaussian map μ is injective for the general curve of genus $g \leqslant 10$.* □

4. Binary curves: the torsion

Let $C = C_1 \cup C_2$ be a general binary curve as in §2. In (4) we may replace f_k, $1 \leqslant k \leqslant 2$, with

$$A_k(t) f_k(t) = [\phi_{k,1}(t), \ldots, \phi_{k,g}(t)], \qquad \phi_{k,i}(t) = \frac{A_k(t)}{(t-\alpha_{k,i})}. \tag{15}$$

Now we consider the composition of the 2nd Gaussian map for C with the projection on the torsion part T of $H^0(C, \mathrm{Sym}^2(\Omega_C^1)\otimes K_C^{\otimes 2})$, cf. formula (3). By taking into account Lemma 2, (a), a direct computation shows that the composition of μ with the projection on the torsion part T_{p_h} at the coordinate point p_h is as follows: if $Q \in \mathcal{I}_2(K_C)$ is of the form (6), with the s_{ij}'s satisfying (13), then Q is mapped to

$$dx\,dy \sum_{i<j} s_{ij}\phi'_{1,i}(\alpha_{1,h})\phi'_{2,j}(\alpha_{2,h}) + x\,dx\,dy \sum_{i<j} s_{ij}\phi''_{1,i}(\alpha_{1,h})\phi'_{2,j}(\alpha_{2,h}) + y\,dx\,dy \sum_{i<j} s_{ij}\phi'_{1,i}(\alpha_{1,h})\phi''_{2,j}(\alpha_{2,h}), \tag{16}$$

where x, y are local coordinates around p_h such that $C_1\colon x=0$ and $C_2\colon y=0$. The description of the torsion at the unitary point u is similar. Replace f_k by the parametrization $\frac{1}{t}\, f_k(\frac{1}{t})$. Again a direct computation shows that the composition of μ with the projection on T_u is

$$Q \mapsto dx\,dy \sum_{i<j} s_{ij}\alpha_{1,i}\alpha_{2,i} + x\,dx\,dy \sum_{i<j} s_{ij}\alpha_{1,i}^2\alpha_{2,i} + y\,dx\,dy \sum_{i<j} s_{ij}\alpha_{1,i}\alpha_{2,i}^2, \tag{17}$$

where x, y are local coordinates around u such that $C_1\colon x=0$ and $C_2\colon y=0$.

Consider the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & T & \longrightarrow & H^0(C, \mathrm{Sym}^2(\Omega_C^1)\otimes K_C^{\otimes 2}) & \twoheadrightarrow & H^0(C_1, K_{C_1}^{\otimes 2}(2)) \oplus H^0(C_2, K_{C_1}^{\otimes 2}(2)) \\ & & \uparrow{\scriptstyle \tau} & & \uparrow{\scriptstyle \mu} & \nearrow{\scriptstyle \nu} & \\ 0 & \longrightarrow & \ker(\nu) & \longrightarrow & \mathcal{I}_2(C) & & \end{array}$$

We asked Maple to compute the rank of the map τ for $11 \leqslant g \leqslant 18$ (see the script in the Appendix). Taking into account the above diagram, the result is the following:

Proposition 11. *Let C be a general binary curve of genus g. The maps τ and μ have maximal rank for $g \leqslant 18$, namely they are injective for $g \leqslant 17$ and surjective for $g = 18$.* □

Corollary 12. *The map μ is injective for the general curve of genus $g \leqslant 17$, and surjective for $g = 18$.* □

5. The induction step

In this section we prove the main result of this paper, namely the surjectivity of the 2nd Gaussian map μ for the general curve of genus $\geqslant 18$.

Let $C \subset \mathbb{P}^{g-1}$ be a nodal canonical curve and let $p \in C$ be a node. Let $\tilde{C} \to C$ be the partial normalization of C at p, and let $p_1, p_2 \in \tilde{C}$ be the points over p. Note that the projection from p maps C to the canonical model of $\tilde{C}$ in $\mathbb{P}^{g-2}$ and we will assume that this induces an isomorphism of $\tilde{C}$ to its canonical model.

Then we have the commutative diagram

$$\begin{array}{ccccc}
\mathcal{I}_2(K_{\tilde{C}}) & \xrightarrow{\tilde{\mu}} & H^0(\mathrm{Sym}^2(\Omega^1_{\tilde{C}}) \otimes K_{\tilde{C}}^{\otimes 2}) & \twoheadrightarrow & H^0(\mathcal{F}_{\tilde{C}}) \\
\downarrow & & & & \downarrow \\
\mathcal{I}_2(K_C) & \xrightarrow{\mu} & H^0(\mathrm{Sym}^2(\Omega^1_C) \otimes K_C^{\otimes 2}) & \twoheadrightarrow & H^0(\mathcal{F}_C) \\
 & \searrow^{\chi} & & & \downarrow \\
 & & & & \mathcal{O}_{2p_1} \oplus \mathcal{O}_{2p_2}
\end{array}$$

(with ν the arrow from $\mathcal{I}_2(K_C)$ to $H^0(\mathcal{F}_C)$)

The following lemma comes from a standard diagram chase:

Lemma 13. *If $\tilde{\mu}$ and χ are surjective, then μ is also surjective.* □

We apply this to prove:

Theorem 14. *If $C = C_1 \cup C_2$ is a general binary curve of genus $g \geqslant 18$, then μ is surjective for C.*

Proof. The case $g = 18$ has been done by a direct computation, cf. Proposition 11. We then proceed by induction applying the previous lemma. Since $\tilde{\mu}$ is surjective by induction, it remains to prove the surjectivity of χ, where p is a node of C. We will do this for $p = u$ the unitary point.

In this situation, the map ν is the one $\nu_1 \oplus \nu_2$ considered in §3. Therefore $\chi = \chi_1 \oplus \chi_2$, where χ_k is the composition of ν_k with the restriction to $\mathcal{O}_{2p_k}$, $k = 1, 2$. In local coordinates, $\chi_k(Q)$ is the pair formed by the constant term and the coefficient of the degree-one term of the Taylor expansion around p of the polynomial $\nu_k(Q)$. In §3 we computed ν_k using a local coordinate t on C_k. In this coordinate, the point $p = [1, \ldots, 1]$ corresponds to $t = \infty$. Therefore, if $Q \in \mathcal{I}_2(K_C)$ is of the form (6), with the s_{ij}'s satisfying (13), then $\chi_k(Q)$ is the pair of coefficients of the highest degrees $2g-6$ and $2g-7$ of the polynomial $\nu_k(Q)$, i.e. of the polynomial $R_k(t)$ given in (14). We denote by $R_{k,2g-6}$ and $R_{k,2g-7}$ these coefficients, which are linear polynomials in the s_{ij}'s. We will now compute them.

Fix the index k and omit it. By expanding A^2 in (14), one sees that the coefficient of s_{ij} in R_{2g-6} is

$$4p_{2;i,j} + \sum_{i_1 \neq i,j} \alpha_{i_1}^2 = 4p_{2;i,j} + n_2 - (\alpha_i^2 + \alpha_j^2),$$

where $n_2 = \sum_{m=1}^g \alpha_m^2$ is independent of i, j, and $p_{2;i,j}$ is the coefficient of s_{ij} in $P_{k,g-4}$, cf. (7). By (10), this means that

$$R_{2g-6} = 4P_{g-4} + n_2 P_{g-2} - \sum_{i<j} (\alpha_i^2 + \alpha_j^2) s_{ij} = -\sum_{i<j} (\alpha_i^2 + \alpha_j^2) s_{ij}.$$

Similarly, one sees that the coefficient of $s_{i,j}$ in R_{2g-7} is twice

$$-4p_{3;i,j} - \sum_{\substack{i_1 \neq i_2 \\ \text{both} \neq i,j}} \alpha_{i_1}^2 \alpha_{i_2} = -4p_{3;i,j} + n_3 + n_2 q_{1;i,j} - c_1(\alpha_i^2 + \alpha_j^2) - (\alpha_i^3 + \alpha_j^3) - q_{3;i,j}.$$

where $n_3 = -\sum_{m=1}^g \alpha_m^3$ is independent of i, j. Therefore, taking into account (10) and (11), one has

$$R_{2g-7} = -2c_1 R_{2g-6} - 2\sum_{i<j} (\alpha_i^3 + \alpha_j^3) s_{ij}.$$

Form the matrix $Y := Y(\alpha_{k,i})_{1 \leqslant k \leqslant 2, 1 \leqslant i < j \leqslant g}$ of size $(2g+1) \times \binom{g}{2}$ obtained by concatenating vertically the matrix Z in the proof of Proposition 7, (b), and the matrix of size $4 \times \binom{g}{2}$ whose rows are $(\alpha_{k,i}^h + \alpha_{k,j}^h)_{1 \leqslant i < j \leqslant g}$, with $1 \leqslant k \leqslant 2$, $2 \leqslant h \leqslant 3$. In order to accomplish the proof of the theorem we have to prove that there is a minor of order $2g+1$ of Y which is not identically zero. We will do this for the minor $N := N(\alpha_{k,i})_{1 \leqslant k \leqslant 2, 1 \leqslant i < j \leqslant g}$ determined by the first $2g+1$ columns, indexed by $(1, i)$, $(2, j)$, $(3, \ell)$,

with $2 \leqslant i \leqslant g$, $3 \leqslant j \leqslant g$, $4 \leqslant \ell \leqslant 7$. This is non-zero for $g = 7$: we verified this with Maple (see the script in the Appendix). Then we proceed by induction on g and we assume $g \geqslant 8$. The argument here is the same as the one in the proof of Proposition 7, (b). Look at N as a polynomial in $\alpha_{1,g}$ and $\alpha_{2,g}$: one sees that N has degree $g-2$ in both $\alpha_{1,g}$, $\alpha_{2,g}$ and the monomials of the highest degree are $\alpha_{k,g}^{g-2}\alpha_{3-k,g}^{g-3}$, $1 \leqslant k \leqslant 2$, whose coefficient is $(-1)^k(\alpha_{3-k,1} - \alpha_{3-k,2})N(\alpha_{k,i})_{1\leqslant k\leqslant 2, 1\leqslant i<j\leqslant g-1}$, which is non-zero by induction, proving the assertion. □

Corollary 15. *The 2nd Gaussian map μ is surjective for the general curve of genus $g \geqslant 18$.* □

Appendix: Maple scripts for computations

We list here the Maple script we run. We will explain it afterwards: for this purpose, we added line numbers at each five lines.

```
alpha[1]:=[1,3,5,7,9,11,13,15,17,19,21,23,25,27,29,31,33,35]:
alpha[2]:=[2,4,6,10,12,14,16,20,22,24,26,28,30,32,34,36,38,40]:
for g from 4 to 18 do
listsij:=[seq(seq(s[i,j],j=i+1..g),i=1..g)]:
for k from 1 to 2 do
  A[k]:=mul(t-alpha[k][i],i=1..g):
  R[k]:=add(add(s[i,j]*(A[k]^2)/((t-alpha[k][i])^2*(t-alpha[k][j])^2),
                j=i+1..g),i=1..g):
end do:
Z:=linalg[matrix]([seq([seq(seq(add(alpha[1][i]^m*alpha[1][j]^(h-m),m=0..h),
                   j=i+1..g),i=1..g)],h=0..g-2),
                   seq([seq(seq(add(alpha[2][i]^m*alpha[2][j]^(h-m),m=0..h),
                   j=i+1..g),i=1..g)],h=1..g-2)]):
Zref:=Gausselim(Z,'r0') mod 41:
printf("For g=%2d, one has dim I2(K)=%3d, ",g,nops(listsij)-r0):
EqsKerNu:=[seq(seq(primpart(coeff(R[k],t,n)),n=0..2*g-6),k=1..2)]:
K:=Gausselim(linalg[stackmatrix](Zref,
             linalg[genmatrix](EqsKerNu,listsij)),'r1') mod 41:
printf("dim Ker(nu)=%2d, corank(nu)=%d, ",nops(listsij)-r1,4*g-10-r1+r0):
for k from 1 to 2 do for i from 1 to g do
  phi1[k,i]:=diff(A[k]/(t-alpha[k][i]),t): phi2[k,i]:=diff(phi1[k,i],t):
  for h from 1 to g do
    phi1e[k,i,h]:=eval(phi1[k,i],t=alpha[k][h]):
    phi2e[k,i,h]:=eval(phi2[k,i],t=alpha[k][h]):
end do: end do: end do:
for h from 1 to g do
  tors[h,1]:=add(add(s[i,j]*phi1e[1,i,h]*phi1e[2,j,h],j=i+1..g),i=1..g):
  tors[h,2]:=add(add(s[i,j]*phi2e[1,i,h]*phi1e[2,j,h],j=i+1..g),i=1..g):
  tors[h,3]:=add(add(s[i,j]*phi1e[1,i,h]*phi2e[2,j,h],j=i+1..g),i=1..g):
end do:
tors[0,1]:=add(add(s[i,j]*alpha[1][i]*alpha[2][j],j=i+1..g),i=1..g):
tors[0,2]:=add(add(s[i,j]*alpha[1][i]^2*alpha[2][j],j=i+1..g),i=1..g):
tors[0,3]:=add(add(s[i,j]*alpha[1][i]*alpha[2][j]^2,j=i+1..g),i=1..g):
EqsKerTau:=[seq(seq(primpart(tors[h,l]),l=1..3),h=0..g)]:
Gausselim(linalg[stackmatrix](K,linalg[genmatrix](EqsKerTau,listsij)),'r2') mod 41:
printf("dim ker(tau)=%d, corank(tau)=%2d\n",nops(listsij)-r2,3*g+3-r2+r1):
if g=7 then
  N:=linalg[det](linalg[stackmatrix](linalg[delcols](Z,16..21),
                 linalg[matrix]([seq(seq([seq(seq(alpha[k][i]^h+alpha[k][j]^h,
                                 j=i+1..7),i=1..3)],h=2..3),k=1..2)]))):
  printf("For g= 7, the minor N is congruent to %d (mod 5)\n",N mod 5):
end if:
end do:
```

In lines 1–2, we define the $\alpha_{k,i}$'s which will be used. In line 3, we start the main loop which goes for $4 \leqslant g \leqslant 18$. In line 4, we collect the unknowns $\{s_{i,j}\}_{1 \leqslant i < j \leqslant g}$ in the list `listsij`: there are $\binom{g}{2}$ of them. In lines 6–8 we define the polynomials $A_k(t)$ and $R_k(t)$, cf. (5) and (14).

In lines 10–13, we define the matrix `Z` associated to the linear system (13), whose solutions give us the quadrics in $\mathcal{I}_2(K_C)$, cf. the proof of Proposition 7. In line 14, Maple computes the rank `r0` of `Z` via Gaussian elimination, by calculating modulo 41. The resulting matrix in row echelon form is called `Zref`. As expected by Proposition 7, (b), Maple finds $\texttt{r0} = 2g-3$ for each $g = 4, \ldots, 18$. In line 15, Maple prints out the genus g and $\dim(\mathcal{I}_2(K_C)) = \binom{g}{2} - \texttt{r0} = \binom{g-2}{2}$.

In line 16, we collect in `EqsKerNu` the list of equations which determine $\ker(\nu)$, cf. the definition (14) of ν in §3. In lines 17–18, Maple computes the rank `r1` of the linear system $\texttt{EqsKerNu} \cap \ker(\texttt{Zref})$, again via Gaussian elimination modulo 41, and the resulting row echelon matrix is called `K`. Maple finds that $\texttt{r1} = \binom{g}{2}$ for $4 \leqslant g \leqslant 10$ and that $\texttt{r1} = 6g-13$ for $11 \leqslant g \leqslant 18$. Therefore the rank of ν is $\texttt{r1} - \texttt{r0} = \binom{g-2}{2}$ for $4 \leqslant g \leqslant 10$, and $= 4g-10$ for $11 \leqslant g \leqslant 18$. This proves Proposition 9.

In line 19, Maple prints out the dimension of $\ker(\nu)$ and the corank of ν, that is $4g - 10 - \texttt{r1} + \texttt{r0}$.

In lines 20–25, we define the 1st derivative `phi1` and the 2nd one `phi2` of the $\phi_{k,i}$'s, cf. (15). We then define their evaluations `phi1e`, `phi2e` at the coordinate point p_h. Using them, in lines 26–30 we compute the torsion at p_h, $h = 1, \ldots, g$, cf. (16), and, in lines 31–33, the torsion at the unit point u, cf. (17).

In lines 34–35, we collect in `EqsKerTau` the equations which determine $\ker(\tau)$ and Maple computes the rank `r2` of $\texttt{EqsKerTau} \cap \ker(\texttt{K})$, via Gaussian elimination modulo 41 as before. Maple finds that $\texttt{r2} = \binom{g}{2}$ for $4 \leqslant g \leqslant 17$ and that $\texttt{r2} = 152$ for $g = 18$. Therefore the rank of τ is $\texttt{r2} - \texttt{r1} = (g^2 - 13g + 26)/2$ for $11 \leqslant g \leqslant 17$, and $= 57$ for $g = 18$. This proves Proposition 11.

In line 36, Maple prints out the the dimension of $\ker(\tau)$ and the corank of τ, that is $3g + 3 - \texttt{r2} + \texttt{r1}$.

Finally, in lines 37–42, when $g = 7$, Maple computes the minor N needed in the proof of Theorem 14 and prints out $N \pmod 5$, which turns out to be 1.

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E-mail address: `calabri@dmsa.unipd.it`

Current address: DMMMSA, Università di Padova, Via Trieste 63, 35121 Padova, Italy

E-mail address: `cilibert@mat.uniroma2.it`

Current address: Dipartimento di Matematica, Università di Roma Tor Vergata, Via della Ricerca Scientifica, 00133 Roma, Italy

E-mail address: `Rick.Miranda@ColoState.edu`

Current address: Department of Mathematics, 101 Weber Building, Colorado State University, Fort Collins, CO 80523-1874, USA