MATH261 EXAM I SPRING 2008

NAME: Key

SI: ________________

SECTION NUMBER: _____

Problem	Points	Score
1	15	
2	10	
3	10	
4	8	
5	12	
6	15	
7	15	
8	15	
Total	100	

You may NOT use calculators or any references. Show work to receive full credit.

GOOD LUCK !!!

1. Let $\mathbf{u} = \langle 5, 0, -1\rangle$ and $\mathbf{v} = \langle -2, 3, 1\rangle$.

   (a) Let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$. Find $\cos\theta$ and determine if the angle is acute or obtuse.

   (b) Find $\text{proj}_{\mathbf{v}}\mathbf{u}$

   (c) Find a vector orthogonal to $\mathbf{v}$ where $\mathbf{u}$ is the sum of this vector and $\text{proj}_{\mathbf{v}}\mathbf{u}$.

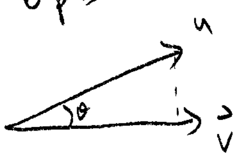
   (d) Verify that the vectors from (b) and (c) are orthogonal.

4 pts (a)

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|} = \frac{-10+0-1}{\sqrt{26}\,\sqrt{14}} = \frac{-11}{2\sqrt{91}}$$ ✱

θ is obtuse

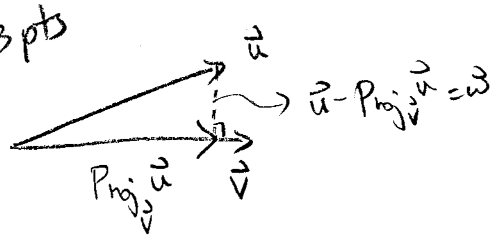
6 pts (b)



$$\text{Proj}_{\vec{v}}\vec{u} = |\vec{u}|\cos\theta\cdot\frac{\vec{v}}{|\vec{v}|} = \frac{\cancel{|\vec{u}|}(\vec{u}\cdot\vec{v})\,\vec{v}}{\cancel{|\vec{u}|}\,|\vec{v}|^2} = \left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\right)\vec{v}$$

$$= \frac{-11}{(\sqrt{14})^2}\cdot\langle -2, 3, 1\rangle = \frac{-11}{14}\langle -2, 3, 1\rangle$$ ✱

3 pts (c)



$$\vec{w} = \vec{u} - \text{Proj}_{\vec{v}}\vec{u} = \langle 5, 0, -1\rangle - \frac{-11}{14}\langle -2, 3, 1\rangle$$

$$= \left\langle 5 - \frac{11}{7},\ \frac{33}{14},\ -1 + \frac{11}{14}\right\rangle$$

$$= \left\langle \frac{24}{7},\ \frac{33}{14},\ \frac{-3}{14}\right\rangle \overset{\text{or}}{=} \frac{1}{14}\langle 48, 33, -3\rangle$$ #

2 pts (d)

$$\text{Proj}_{\vec{v}}\vec{u}\cdot\vec{w} = \frac{-11}{14}\langle -2, 3, 1\rangle\cdot\frac{1}{14}\langle 48, 33, -3\rangle$$

$$= \frac{-11}{14^2}\Big((-2)(48) + 3(33) + (1)(-3)\Big) = 0$$

$\therefore\ \text{Proj}_{\vec{v}}\vec{u} \perp \vec{w}$.

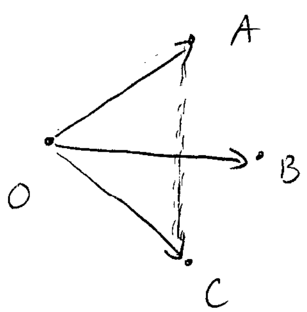
2. Let $O(0,0,0)$, $A(1,2,3)$, $B(2,1,1)$, and $C(1,-1,0)$ be four points.

4pts (a) Find the volume of the parallelpiped constructed from $\overrightarrow{OA}, \overrightarrow{OB}$, and $\overrightarrow{OC}$.

4pts (b) Find the area of the triangle $\triangle OAC$.

1 pt (c) Find a normal vector to the plane defined by the points O, A, C.

1 pt (d) Write the equation for the plane defined by the points O, A, C.



(a)

$$V = \left| \overrightarrow{OA} \times \overrightarrow{OB} \cdot \overrightarrow{OC} \right| = \left| \langle -1, 5, -3 \rangle \cdot \langle 1, -1, 0 \rangle \right|$$

$$= |-1-5+0| = 6$$

✱

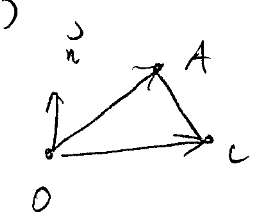
(b) $A(\triangle OAC) = \frac{1}{2} \left| \overrightarrow{OA} \times \overrightarrow{OC} \right|$

$$= \frac{1}{2} 3\sqrt{3} = \frac{3\sqrt{3}}{2}$$

✱

$$\overrightarrow{OA} \times \overrightarrow{OB} = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 2 & 1 & 1 \end{vmatrix} = \langle -1, 5, -3 \rangle$$

(c)



$$\overrightarrow{OA} \times \overrightarrow{OC} = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 1 & -1 & 0 \end{vmatrix} = \langle 3, 3, -3 \rangle$$

$$\left| \overrightarrow{OA} \times \overrightarrow{OC} \right| = \sqrt{3^2 + 3^2 + 3^2} = \sqrt{27} = 3\sqrt{3}$$

$$\vec{n} = \overrightarrow{OA} \times \overrightarrow{OC} = \langle 3, 3, -3 \rangle$$

✱

(d) use $\vec{n} = \langle 3, 3, -3 \rangle$ and $O(0,0,0)$

$$\Pi: \quad 3(x-0) + 3(y-0) - 3(z-0) = 0$$

$$3x + 3y - 3z = 0 \quad \text{or} \quad \boxed{x + y - z = 0}$$

✱

3. Given two planes $P_1 : 3x - 3y + z = 6$ and $P_2 : -x + 4y + z = 2$;

(a) find a vector parallel to the line of intersection.

(b) If $S(1, 0, 3)$ is in the intersection of P_1, P_2, find a parametric equation for the line of intersection going through point S.

(c) Given a point $Q(1, 2, -4)$, determine if Q is closer to P_1 or P_2. You must show all work to receive any credit!

(a) $\vec{n_1} = \langle 3, -3, 1 \rangle$, $\vec{n_2} = \langle -1, 4, 1 \rangle$

+3pts

$$\vec{n_1} \times \vec{n_2} = \begin{vmatrix} i & j & k \\ 3 & -3 & 1 \\ -1 & 4 & 1 \end{vmatrix} = \langle -7, -4, 9 \rangle$$

✱

(b)

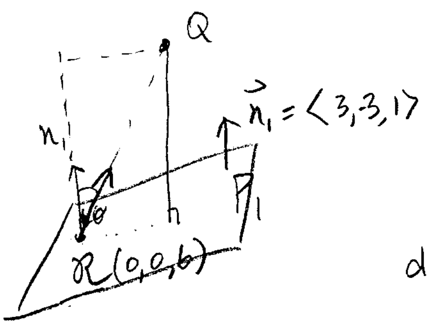
+3pts

$$\begin{cases} x = 1 - 7t \\ y = -4t \\ z = 3 + 9t \end{cases}, \quad t \in \mathbb{R}.$$

✱

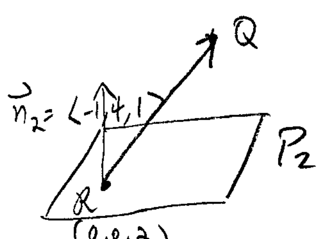
+4 pts (c) $d(Q, P_1) = |\vec{RQ}| \cos\theta = \dfrac{\cancel{|\vec{RQ}|}\, |\vec{RQ} \cdot \vec{n}|}{\cancel{|\vec{RQ}|}\, |\vec{n}|} = \dfrac{|\langle 1, 2, -10 \rangle \cdot \langle 3, -3, 1 \rangle|}{\sqrt{9+9+1}}$

$$= \frac{|3 - 6 - 10|}{\sqrt{19}} = \frac{13}{\sqrt{19}}$$



$$d(Q, P_2) = \frac{|\langle -1, 4, 1 \rangle \cdot \langle 1, 2, -6 \rangle|}{\sqrt{1 + 16 + 1}} = \frac{|-1 + 8 - 6|}{\sqrt{18}} = \frac{1}{\sqrt{18}}$$

$\vec{RQ} = \langle 1, 2, -10 \rangle$



Clearly $d(Q, P_1) > d(Q, P_2)$

$\therefore$ Q is closer than P_2

✱

$\vec{RQ} = \langle 1, 2, -6 \rangle$

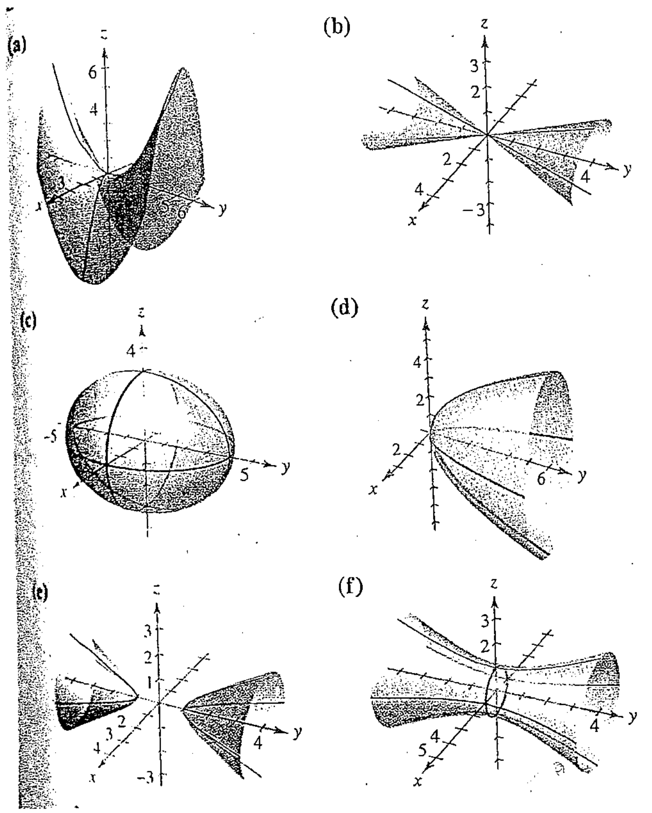
4. Match the following equations with the appropriate graph;

2 pts (P1) $$\frac{x^2}{9} + \frac{y^2}{16} + \frac{z^2}{9} = 1,$$ ellipsoid. (c)

2 pts (P2) $$15x^2 - 4y^2 + 15z^2 = -4.$$ (e)

2 pts (P3) $$y^2 = 4x^2 + 9z^2$$ Cone (b)

2 pts (P4) $$-4x^2 + y^2 = 4z$$ Saddle (a)



P2: $4y^2 - 15x^2 - 15z^2 = 4$

when $x=0$: $4y^2 - 15z^2 = 4$ hyperbola

when $z=0$: $4y^2 - 15x^2 = 4$ "

when $y=0$: $-(15x^2 + 15z^2) = 4$

doesn't make sense

must be 2 sheets.

$\therefore$ (e)

5. Let $\mathbf{r}(t) = 5\sin t\mathbf{i} + 5\cos t\mathbf{j} + 3t\mathbf{k}$ be a position vector.

(a) Find $\mathbf{v}(t)$ and $\mathbf{a}(t)$.

(b) What is the arc length when traveling on the curve from $t = 0$ to $t = 4$.

(c) Find $\kappa(0)$.

(a) $\vec{v}(t) = \langle 5\cos t, -5\sin t, 3 \rangle$ 3 pts

$\vec{a}(t) = \langle -5\sin t, -5\cos t, 0 \rangle$ 3pts

(b)

$|\vec{v}| = \sqrt{25+9} = \sqrt{34}$ 2pts

$L = \int_{t=0}^{t=4} \sqrt{34}\, dt = 4\sqrt{34}$ * 1pt

(c) $\kappa = \dfrac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3}$ $\kappa(0) = \dfrac{|\vec{v}(0) \times \vec{a}(0)|}{|\vec{v}(0)|^3} = \dfrac{\sqrt{15^2 + 25^2}}{(\sqrt{34})^3}$

$\vec{v}(0) = \langle 5, 0, 3 \rangle$ $= \dfrac{\sqrt{850}}{34\sqrt{34}}$ * 3 pts

$\vec{a}(0) = \langle 0, -5, 0 \rangle$

$$\vec{v}(0) \times \vec{a}(0) = \begin{vmatrix} i & j & k \\ 5 & 0 & 3 \\ 0 & -5 & 0 \end{vmatrix}$$

$= \langle 15, 0, -25 \rangle$ 3pts

6. Given $\mathbf{r}(t) = (t^2 - t)\mathbf{i} + t^3\mathbf{j} + t^2\mathbf{k}$, $\mathbf{v}(t) = (2t-1)\mathbf{i} + 3t^2\mathbf{j} + 2t\mathbf{k}$, and $\mathbf{a}(t) = 2\mathbf{i} + 6t\mathbf{j} + 2\mathbf{k}$.

6 pts (a) Find $\mathbf{r}$, $\mathbf{v}$, $\mathbf{a}$ at $t = 2$,

4 pts (b) find $\mathbf{T}$ at $t = 2$,

3 pts (c) find a_T at $t = 2$,

2 pts (d) find a_N at $t = 2$.

(a) $\vec{r}(2) = \langle 2, 8, 4\rangle$

$\vec{v}(2) = \langle 3, 12, 4\rangle$

$\vec{a}(2) = \langle 2, 12, 2\rangle$

(b) $\vec{T} = \dfrac{\vec{v}}{|\vec{v}|}$, $\vec{T}(2) = \dfrac{\vec{v}(2)}{|\vec{v}|_{t=2}} = \dfrac{\langle 3, 12, 4\rangle}{\sqrt{9+144+16}} = \frac{1}{13}\langle 3, 12, 4\rangle$

$= \left\langle \frac{3}{13}, \frac{12}{13}, \frac{4}{13} \right\rangle$ #

(c) $a_T = \frac{d}{dt}|\vec{v}|$, $a_N = \kappa|\vec{v}|^2$

$$\kappa(2) = \frac{|\vec{v}(2) \times \vec{a}(2)|}{|\vec{v}(2)|^3} = \frac{\sqrt{24^2+2^2+12^2}}{13^3} = \frac{\sqrt{724}}{13^3} = \frac{2\sqrt{181}}{13^3}$$

$$\vec{v}(2) \times \vec{a}(2) = \begin{vmatrix} i & j & k \\ 3 & 12 & 4 \\ 2 & 12 & 2 \end{vmatrix}$$

$= \langle -24, 2, 12\rangle$

$\therefore a_N = \frac{2\sqrt{181}}{13^3} \cdot 13^2 = \frac{2}{13}\sqrt{181}$

$|\vec{a}(2)| = \sqrt{4+4+144} = \sqrt{152}$

$\therefore a_T = \sqrt{152 - \frac{4}{169}(181)} = \sqrt{152 - \frac{724}{169}}$

$= \sqrt{\frac{24964}{169}} = \frac{158}{13}$ #

(d) $a_N = \frac{2}{13}\sqrt{181}$ #

7. Suppose a fly starts at the position (1,-2,-1) and has an initial velocity of $\mathrm{v} = \langle 1, 0, 2\rangle$. Assume its acceleration fits the vector function

$$\mathrm{a}(t) = \langle e^{2t}, \cos t, \sqrt{t+1}\rangle.$$

Find the position vector $\mathrm{r}(t)$.

$\vec{r}(0) = (1, -2, -1)$

$\vec{V}(0) = \langle 1, 0, 2\rangle$

$\vec{V}(t) = \int \vec{a}\, dt = \left\langle \int e^{2t} dt, \int \cos t\, dt, \int \sqrt{t+1}\, dt \right\rangle$

$= \left\langle \frac{1}{2}e^{2t}, \sin t, \frac{2}{3}(t+1)^{\frac{3}{2}} \right\rangle + \vec{C_1}$

$\vec{V}(0) = \left\langle \frac{1}{2}, 0, \frac{2}{3} \right\rangle + \vec{C_1} = \langle 1, 0, 2\rangle$

$\Rightarrow \vec{C_1} = \left\langle \frac{1}{2}, 0, \frac{4}{3} \right\rangle$

$\therefore \vec{v}(t) = \left\langle \frac{1}{2}e^{2t} + \frac{1}{2}, \sin t, \frac{2}{3}(t+1)^{\frac{3}{2}} + \frac{4}{3} \right\rangle$ 6 pts

3 pts $\vec{r}(t) = \int \vec{v}(t)\, dt = \left\langle \frac{1}{4}e^{2t} + \frac{1}{2}t, -\cos t, \frac{2}{3}\cdot\frac{2}{5}(t+1)^{\frac{5}{2}} + \frac{4}{3}t \right\rangle + \vec{C_2}$

$\vec{r}(0) = \left\langle \frac{1}{4}, -1, \frac{4}{15} \right\rangle + \vec{C_2} = \langle 1, -2, -1\rangle$

3 pts $\Rightarrow \vec{C_2} = \left\langle \frac{3}{4}, -1, -\frac{19}{15} \right\rangle$

$\therefore \vec{r}(t) = \left\langle \frac{1}{4}e^{2t} + \frac{1}{2}t + \frac{3}{4}, -\cos t - 1, \frac{4}{15}(t+1)^{\frac{5}{2}} + \frac{4}{3}t - \frac{19}{15} \right\rangle$

3 pts

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8. Give the value of the limit or show that the following limit does not exist:

4 pt (a)

$$\lim_{(x,y)\to(1,1)} \frac{2xy}{x^2+3xy+2y^2}$$

5 pt (b)

$$\lim_{(x,y)\to(1,-1)} \frac{2xy}{x^2+3xy+2y^2}$$

6 pt (c)

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+3xy+2y^2}$$

let $F(x,y) = \dfrac{2xy}{x^2+3xy+2y^2}$, $p=(x,y)$

(a) $$\lim_{p\to(1,1)} F(x,y) = \frac{2(1)(1)}{1^2+3(1)(1)+2(1)^2} = \frac{2}{1+3+2} = \frac{2}{6} = \frac{1}{3}$$

(b) $$\lim_{p\to(1,-1)} F(x,y) = \frac{2(1)(-1)}{1^2+3(1)(-1)+2(-1)^2} = \frac{-2}{0} = \text{undefined.}$$

(c) $$\lim_{p\to(0,0)} F(x,y) = \lim_{p\to(0,0)} \frac{2x(mx)}{x^2+3x(mx)+2(mx)^2} = \lim_{p\to(0,0)} \frac{2m\cancel{x^2}}{(1+3m+2m^2)\cancel{x^2}}$$

(Consider the path $y = mx$)

$$= \lim_{p\to(0,0)} \frac{2m}{1+3m+2m^2}$$

when $m = 0$, $\lim\limits_{p\to(0,0)} F = 0$

when $m = 1$, $\lim\limits_{p\to(0,0)} F = \dfrac{2}{6} = \dfrac{1}{3}$

Since $0 \neq \frac{1}{3}$

$\therefore \lim\limits_{p\to(0,0)} F(x,y)$ DNE

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