The Lowest-order Weak Galerkin Finite Element Method for the Darcy Equation on Quadrilateral and Hybrid Meshes

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Abstract

This paper presents the lowest-order weak Galerkin finite element method for solving the Darcy equation on quadrilateral and hybrid meshes consisting of quadrilaterals and triangles. In this approach, the pressure is approximated by constants in element interiors and on edges. The discrete weak gradients of these constant basis functions are specified in local Raviart-Thomas spaces $RT_0$ for quadrilaterals or $RT_0$ for triangles. These discrete weak gradients are used to approximate the classical gradient when solving the Darcy equation. The method produces continuous normal fluxes and is locally mass-conservative, regardless of mesh quality. It exhibits expected convergence in pressure, velocity, and flux when the quadrilaterals are asymptotically parallelograms. Implementation is straightforward and results in symmetric positive-definite discrete linear systems. Numerical experiments and comparison with other existing methods are also presented.

Key words: Darcy equation, hybrid meshes, lowest order finite elements, quadrilateral meshes, weak Galerkin

Preprint submitted to Elsevier
1. Introduction

This paper concerns with finite element methods for boundary value problems of the Darcy equation

\[
\begin{align*}
\nabla \cdot (-K \nabla p) &\equiv \nabla \cdot \mathbf{u} = f, & \mathbf{x} \in \Omega, \\
p &\equiv p_D, & \mathbf{x} \in \Gamma^D, \\
\mathbf{u} \cdot \mathbf{n} &\equiv u_N, & \mathbf{x} \in \Gamma^N,
\end{align*}
\]

(1)

where \( \Omega \subset \mathbb{R}^2 \) is a bounded (polygonal) domain, \( p \) the unknown pressure, \( K \) a permeability matrix that is uniformly symmetric positive-definite, \( f \) a source term, \( p_D, u_N \) respectively Dirichlet and Neumann boundary data, \( \mathbf{n} \) the outward unit normal vector on \( \partial \Omega \), which has a nonoverlapping decomposition \( \Gamma^D \cup \Gamma^N \).

Efficient and robust solvers for the Darcy equation are fundamentally important for numerical simulations of flow and transport in porous media. Some early work can be found in [14, 15, 30]. The two most important properties desired for Darcy solvers are local mass-conservation and normal flux continuity. For the continuous Galerkin (CG) methods, these can be achieved by postprocessing [13]. The discontinuous Galerkin methods are locally mass-conservative by design, and postprocessing will provide normal flux continuity [7]. The mixed finite element methods (MFEMs) have both properties by design [8], but indefinite linear systems need to be solved. The enriched Galerkin methods attain these properties by enriching the CG approximation spaces with discontinuous piecewise constants [31]. Other research efforts can be found in [9].

Development of Darcy solvers on quadrilateral meshes is more challenging than those for rectangular and triangular meshes. In [4, 5, 34], the Piola transformation is utilized to maintain normal flux continuity on general quadrilateral meshes.

The recently development weak Galerkin (WG) finite element methods [32] bring in new perspectives. When applied to the Darcy equation on triangular and rectangular meshes [22, 23], the WG methods satisfy the aforementioned two physical properties, have optimal order convergence rates, and result in symmetric positive-definite discrete linear systems. Under certain conditions [20], the WG methods satisfy the discrete maximum principle.

However, Darcy solvers on rectangular meshes have limited use for real applications. In certain cases, quadrilateral meshes or hybrid meshes consist-
ing of quadrilaterals and triangles are desired to accommodate complicated domain geometry while keeping degrees of freedom low [1, 10, 37]. WG methods have been developed for the elliptic equation on general polygonal meshes [28, 27]. These methods can be applied to solve the Darcy equation on quadrilateral and triangular meshes. But these WG methods involve a stabilization parameter, for which an optimal value is generally unknown. Although these WG methods offer a continuous normal flux, but it is unclear how a numerical velocity can be calculated.

In this paper, we develop a new WG method for solving the Darcy equation on quadrilateral and hybrid meshes. This new method uses the lowest order approximants for pressure, namely, just constants inside elements and on edges. But the method is locally conservative and produces continuous normal fluxes, regardless of mesh quality. When the quadrilaterals are asymptotically parallelograms, this method has optimal order convergence in pressure, velocity, and flux.

The rest of this paper is organized as follows. In Section 2, we discuss mainly construction of WG\((Q_0, Q_0; RT_{[0]})\) finite elements on quadrilaterals. Section 3 presents the Darcy solver using the lowest order WG finite elements: WG\((Q_0, Q_0; RT_{[0]})\) for quadrilaterals and WG\((P_0, P_0; RT_{[0]})\) for triangles. The properties and errors of this Darcy solver are discussed in Section 4. Section 5 briefly discusses two other finite element methods developed for Darcy problems, a mixed method based on the Piola transformation [5] and a WG method with stabilization [27]. (Recent work [3] constructs H(div) mixed finite elements on quadrilaterals with minimal use of the Piola transformation, but this new method is not implemented here.) Section 6 presents numerical experiments on this new WG method and comparison to the aforementioned two other methods. Section 7 concludes the paper with some remarks.

2. Lowest-order WG Elements on Quadrilaterals and Triangles

WG \((P_0, P_0; RT_{[0]})\) finite elements on triangles and WG \((Q_0, Q_0; RT_{[0]})\) finite elements on rectangles have been used for solving the Darcy equation in [22]. In this section, we construct WG \((Q_0, Q_0; RT_{[0]})\) finite elements for quadrilaterals. These will include the rectangular WG \((Q_0, Q_0; RT_{[0]})\) elements as a special case. This WG approach is different than the approach used in [5] which employs the Piola transformation to construct mixed finite elements. The mixed finite element methods enforce normal flux continuity at the level of finite element spaces. The WG approach in this paper
uses a larger finite element space for gradients, but normal flux continuity is attained through the bilinear form and properties of discrete weak gradients.

2.1. WG\((Q_0, Q_0; RT_{[0]})\) Finite Elements on Quadrilaterals

Let \(E\) be a convex quadrilateral with center \((x_c, y_c)\). Let \(X = x - x_c, Y = y - y_c\). We denote

\[
\begin{align*}
\mathbf{w}_1 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, & \mathbf{w}_2 &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}, & \mathbf{w}_3 &= \begin{bmatrix} X \\ 0 \end{bmatrix}, & \mathbf{w}_4 &= \begin{bmatrix} 0 \\ Y \end{bmatrix},
\end{align*}
\]

and define a local Raviart-Thomas space as

\[
RT_{[0]}(E) = \text{Span}(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4).
\]

The Gram matrix of the above basis is obviously

\[
GM = \begin{bmatrix} |E| & 0 & \int_E X & 0 \\ 0 & |E| & 0 & \int_E Y \\ \int_E X & 0 & \int_E X^2 & 0 \\ 0 & \int_E Y & 0 & \int_E Y^2 \end{bmatrix},
\]

where \(|E|\) is the quadrilateral area. This SPD matrix becomes a diagonal matrix, when \(E\) is a rectangle. But in general, the integrals \(\int_E X, \int_E Y\) are nonzero.

Now we consider five discrete weak functions \(\phi_0, \phi_1, \phi_2, \phi_3, \phi_4\) on \(E\) and assume,

- \(\phi_0\) takes value 1 in the element interior but is 0 on all four edges,
- \(\phi_i, i = 1, 2, 3, 4\) each take value 1 on a single edge but have value 0 on the other three edges and in the element interior.

Note that \(\phi_i = \{\phi_i^o, \phi_i^\partial\}, i = 0, 1, \ldots, 4\), i.e., each of the above five discrete weak functions has two independent components, \(\phi_i^o\) for element interior \(E^o\), and \(\phi_i^\partial\) for element boundary \(E^\partial\).

The discrete weak gradient \(\nabla_{w,d}\phi_i, i = 0, 1, \ldots, 4\) is specified in \(RT_{[0]}(E)\) via integration by parts,

\[
\int_E (\nabla_{w,d}\phi_i) \cdot \mathbf{w} = \int_{E^\partial} \phi_i^\partial (\mathbf{w} \cdot \mathbf{n}) - \int_{E^o} \phi_i^o (\nabla \cdot \mathbf{w}), \quad \forall \mathbf{w} \in RT_{[0]}(E).
\]

4
For $i = 0, 1, \ldots, 4$, let
\[
\nabla_{w,d} \phi_i = c_{i1}w_1 + c_{i2}w_2 + c_{i3}w_3 + c_{i4}w_4,
\]
(6)
where $w_j, \ j = 1, 2, 3, 4$ are defined in (2). For each $i = 0, 1, \ldots, 4$, the coefficients $c_{ij}, j = 1, \ldots, 4$ can be obtained by solving a $4 \times 4$ SPD linear system (5) with coefficient matrix (4). For the special case when $E$ is a rectangle $[x_1, x_2] \times [y_1, y_2]$, and the edges 1 through 4 are bottom, right, top and left respectively, we have [17]
\[
\begin{align*}
\nabla_{w,d} \phi_0 &= 0w_1 + 0w_2 + \frac{-12}{(x_2-x_1)^2}w_3 + \frac{-12}{(y_2-y_1)^2}w_4; \\
\nabla_{w,d} \phi_1 &= 0w_1 + \frac{-1}{y_2-y_1}w_2 + 0w_3 + \frac{6}{(y_2-y_1)^2}w_4; \\
\nabla_{w,d} \phi_2 &= \frac{1}{x_2-x_1}w_1 + 0w_2 + \frac{6}{(x_2-x_1)^2}w_3 + 0w_4; \\
\nabla_{w,d} \phi_3 &= 0w_1 + \frac{1}{y_2-y_1}w_2 + 0w_3 + \frac{6}{(y_2-y_1)^2}w_4; \\
\nabla_{w,d} \phi_4 &= \frac{-1}{x_2-x_1}w_1 + 0w_2 + \frac{6}{(x_2-x_1)^2}w_3 + 0w_4.
\end{align*}
\]
(7)

Figure 1: Left: $\text{WG}(Q_0,Q_0;RT_{[0]})$ element on a quadrilateral; Right: $\text{WG}(P_0,P_0;RT_0)$ element on a triangle.

2.2. $\text{WG} \ (P_0, P_0; RT_0)$ Finite Elements on Triangles

Let $T$ be a triangle with vertices $(x_i, y_i)$, $i = 1, 2, 3$ oriented counterclockwise. Let $(x_c, y_c)$ be the element center and let $X = x - x_c, Y = y - y_c$. We
define
\[ w_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad w_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad w_3 = \begin{bmatrix} X \\ Y \end{bmatrix}, \]
and the local Raviart-Thomas space
\[ RT_0(T) = \text{Span}(w_1, w_2, w_3). \]
The Gram matrix of the above basis is a diagonal matrix
\[ \begin{bmatrix} |T| & 0 & 0 \\ 0 & |T| & 0 \\ 0 & 0 & S \end{bmatrix}, \tag{10} \]
where \(|T|\) is the triangle area, and
\[ S = \frac{1}{12} ((x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_1)^2 + (y_1 - y_2)^2 + (y_2 - y_3)^2 + (y_3 - y_1)^2). \]
We consider 4 discrete shape functions \(\phi_0, \phi_1, \phi_2, \phi_3\) on the triangle and assume
\begin{itemize}
  \item \(\phi_0\) takes value 1 in the element interior but is 0 on all three edges,
  \item \(\phi_i, \ i = 1, 2, 3\) each take value 1 on a single edge but have value 0 on the other two edges and in the element interior.
\end{itemize}
The discrete weak gradients, \(\nabla_{w,d}\phi_i, \ i = 0, 1, 2, 3\) are specified in \(RT_0(T)\) using an equivalent definition to (5). Since the Gram matrix (10) is diagonal, each \(3 \times 3\) linear system can be solved immediately and
\[
\begin{align*}
\nabla_{w,d}\phi_0 &= 0w_1 + 0w_2 + \frac{-2|T|}{S}w_3, \\
\nabla_{w,d}\phi_1 &= \frac{y_3 - y_2}{|T|}w_1 + \frac{x_2 - x_3}{|T|}w_2 + \frac{2|T|}{3S}w_3, \\
\nabla_{w,d}\phi_2 &= \frac{y_1 - y_3}{|T|}w_1 + \frac{x_3 - x_1}{|T|}w_2 + \frac{2|T|}{3S}w_3, \\
\nabla_{w,d}\phi_3 &= \frac{y_2 - y_1}{|T|}w_1 + \frac{x_1 - x_2}{|T|}w_2 + \frac{2|T|}{3S}w_3, \tag{11}
\end{align*}
\]
where \(w_i, \ i = 1, 2, 3\) are defined in (8).
3. The Lowest-order WG Scheme for the Darcy Equation

In this section, we present the weak Galerkin finite element scheme for solving the Darcy equation on quadrilateral or hybrid meshes using the lowest-order WG finite elements discussed in the previous section. In other words, the pressure unknowns are just constants in element interiors and constants on edges. The discrete weak gradients of these shape functions are specified in $RT_{[0]}$ for quadrilaterals and $RT_0$ for triangles. For ease of presentation, we focus on the numerical scheme for a quadrilateral mesh.

Let $\mathcal{E}_h$ be a shape-regular quasi-uniform quadrilateral mesh that consists of convex quadrilaterals. Let $\Gamma^D_h$ be the set of all edges on the Dirichlet boundary $\Gamma^D$ and $\Gamma^N_h$ be the set of all edges on the Neumann boundary $\Gamma^N$. Let $S_h$ be the space of discrete shape functions on $\mathcal{E}_h$ that are degree 0 polynomials in element interiors and also degree 0 polynomials on edges, $S^0_h$ be the subspace of functions in $S_h$ that vanish on $\Gamma^D_h$.

3.1. Quadrilateral mesh

We seek $p^h = \{p^h, p^\partial_h\} \in S_h$ such that $p^\partial_h|_{\Gamma^D_h} = Q^\partial_h(p^D)$ (the $L^2$-projection of Dirichlet boundary data into the space of piecewise constants on $\Gamma^D_h$) and

$$A^h(p^h, q) = F(q), \quad \forall q = \{q^\circ, q^\partial\} \in S^0_h,$$

where

$$A^h(p^h, q) := \sum_{E \in \mathcal{E}_h} \int_E K \nabla_{w,d} p^h \cdot \nabla_{w,d} q,$$

and

$$F(q) := \sum_{E \in \mathcal{E}_h} \int_E f q^\circ - \sum_{\gamma \in \Gamma^N_h} \int_{\gamma} u_N q^\partial.$$

The major steps in the assembly of the global stiffness matrix for this new WG finite element scheme on a quadrilateral mesh are as follows.

(i) On each element,

(a) The Gram matrices (4) of the local $RT_{[0]}$ bases are computed.

(b) The coefficients (6) defining the discrete weak gradients of the five WG basis functions are computed.
(c) The $5 \times 5$ element stiffness matrices corresponding to the term $K \nabla_{w,d} p_h \cdot \nabla_{w,d} q$ in (13) are constructed.

(ii) The element stiffness matrices are assembled into the global stiffness matrix.

The overall procedure is similar to those for rectangular and triangular meshes as discussed in [22]. In Matlab, each of the above steps can be implemented mesh-wise, using the strategies discussed in [25], which is much more efficient than looping over the elements. This new WG solver has been included in the Matlab code package DarcyLite, which can be found on the first author’s webpage.

After the numerical pressure $p_h$ is solved from (12), we compute its discrete weak gradient $\nabla_{w,d} p_h$ and then $-K \nabla_{w,d} p_h$. Note that $-K \nabla_{w,d} p_h$ may not be in $RT_{[0]}$. Then we perform a local $L^2$-projection $R_h$ from $L^2(E)^2$ to $RT_{[0]}(E)$ on any quadrilateral element $E$. This way, we obtained an element-wise numerical velocity $u_h$:

$$u_h = R_h(-K \nabla_{w,d} p_h).$$

(15)

Note that this local $L^2$-projection $R_h$ can be omitted when $K$ is a constant diagonal matrix.

3.2. Hybrid mesh

The above lowest-order WG finite element scheme can be easily extended to a hybrid mesh that consists of quadrilaterals and triangles. Note that the pressure unknown is just a constant inside each element, no matter whether it is a quadrilateral or a triangle. The pressure unknown is also a constant on each edge. Then the discrete weak gradient is specified in $RT_{[0]}$ when the element is a quadrilateral or $RT_0$ when the element is a triangle. These discrete weak gradients will be used in (12), which is solved for the numerical pressure $p_h$. Then a numerical velocity can be obtained from (15) by the local $L^2$-projections on quadrilaterals or triangles.

4. Properties and Convergence of the Lowest-order WG Scheme

Theorem 1 (Local mass conservation). Let $E$ be any quadrilateral or triangular element. There holds

$$\int_E f = \int_{E^0} u_h \cdot n.$$  

(16)
Proof. For simplicity, let $E$ be a quadrilateral. In (12), take a test function $q$ so that $q|_{E^\circ} = 1$ but vanishes on all edges and inside all other elements. Then

$$\int_E f = \int_E (K\nabla_w dp_h) \cdot \nabla_w dq$$

$$= \int_E R_h(K\nabla_w dp_h) \cdot \nabla_w dq$$

$$= -\int_E u_h \cdot \nabla_w dq$$

$$= -\int_{E^\partial} q^\partial(u_h \cdot n) + \int_{E^\circ} q^\circ(\nabla \cdot u_h)$$

$$= \int_{E^\partial} \nabla \cdot u_h$$

$$= \int_{E^\circ} u_h \cdot n.$$

The equalities are provided by (12), the definition of projection $R_h$, the definition of the discrete velocity field, the definition of the discrete weak gradient, the definition of the particular test function $q$, and finally the Divergence Theorem on a vector function that is in the local $RT_{[0]}$ ($RT_0$) space. □

Theorem 2 (Continuity of bulk normal fluxes). Let $\gamma$ be an edge shared by two elements $E_1$ and $E_2$ (each could be a quadrilateral or a triangle) and $n_1, n_2$ be the constant outward unit normal vectors on $\gamma$ for $E_1$ and $E_2$ respectively. Then

$$\int_\gamma u_h^{(1)} \cdot n_1 + \int_\gamma u_h^{(2)} \cdot n_2 = 0.$$  \hspace{1cm} (17)

Proof. Consider equation (12) with a test function $q = \{q^\circ, q^\partial\}$ such that $q^\partial = 1$ on $\gamma$, but $q^\partial = 0$ on all other edges and $q^\circ = 0$ in interior of any
quadrilateral or triangular element. Then from (13) and (14),

\[
0 = \int_{E_1} (K \nabla w, d p_h) \cdot \nabla w, d q + \int_{E_2} (K \nabla w, d p_h) \cdot \nabla w, d q \\
= \int_{E_1} R_h (K \nabla w, d p_h) \cdot \nabla w, d q + \int_{E_2} R_h (K \nabla w, d p_h) \cdot \nabla w, d q \\
= \int_{E_1} (-u_h^{(1)}) \cdot \nabla w, d q + \int_{E_1} (-u_h^{(2)}) \cdot \nabla w, d q \\
= -\int_\gamma u_h^{(1)} \cdot n_1 q^\partial + \int_\gamma u_h^{(1)} q^\circ - \int_\gamma u_h^{(2)} \cdot n_2 q^\partial + \int_\gamma u_h^{(2)} q^\circ \\
= -\int_\gamma u_h^{(1)} \cdot n_1 - \int_\gamma u_h^{(2)} \cdot n_2,
\]

using the definition of projection \( R_h \), the definition of the discrete velocity field, the definition of the discrete weak gradient, and the definition of the particular test function \( q \).

Approximation accuracy of a finite element scheme is affected by mesh quality, which can be defined through element diameters and certain angles. Let \( E \) be a quadrilateral, \( \theta_1 \) be the angle between the outward unit normal vectors on two opposite edges, \( \theta_2 \) be the angle for the other two edges. Let \( \sigma_E = \max\{|\pi - \theta_1|, |\pi - \theta_2|\} \) and \( h_E \) be the diameter of \( E \). A quadrilateral mesh \( \mathcal{E}_h \) is asymptotically parallelogram [5], provided that there exists a positive constant \( C \) such that \( \sigma_E / h_E \leq C \) for all \( E \in \mathcal{E}_h \).

Any polygonal domain can be partitioned into a family of asymptotically parallelogram, shape-regular quadrilateral meshes [4] (p. 918). This can be accomplished by starting from any mesh of convex quadrilaterals and performing regular refinement. This implies that asymptotically parallelogram quadrilateral meshes are adequate for practical use.

To measure errors in pressure, velocity, and flux for this new WG finite element scheme, we use the following norms (see also [33, 32]),

\[
\|p - p^\circ_h\|^2 = \sum_{E \in \mathcal{E}_h} \|p - p^\circ_h\|_{L^2(E)}^2, \\
\|u - u_h\|^2 = \sum_{E \in \mathcal{E}_h} \|u - u_h\|_{L^2(E)}^2, \\
\|(u - u_h) \cdot n\|^2 = \sum_{E \in \mathcal{E}_h} \sum_{\gamma \in E^\partial} \frac{|E|}{|\gamma|} \|u \cdot n - u_h \cdot n\|_{L^2(\gamma)}^2.
\]

10
**Proposition (Convergence in pressure, velocity, and flux).** Let $p$ be the exact solution of the Darcy problem (1) and $\mathbf{u} = -K\nabla p$. Assume the exact solution has regularity $p \in H^{1+s}(\Omega)$ and $\mathbf{u} \in H^{s}(\Omega)^2$ for some $s \in (0, 1]$. When the quadrilaterals are asymptotically parallelograms, there holds

$$
\|p - p_h\| \leq Ch, \quad \|\mathbf{u} - \mathbf{u}_h\| \leq Ch^s, \quad \|\mathbf{u} - \mathbf{u}_h \cdot \mathbf{n}\| \leq Ch^s,
$$

(21)

where $C > 0$ is a constant that is independent of the mesh size $h$.

A complete account of convergence analysis involves the following components.

1. Investigation of the approximation properties of the local Raviart-Thomas spaces $RT_{[0]}$ and the effect of quadrilateral mesh quality on approximation capacity.
2. Establishing the connection among local Raviart-Thomas spaces, the global Raviart-Thomas space, and discrete weak gradients.
3. Establishing an error equation for the finite element solution and the projection of the exact pressure solution.
4. Analyzing the low regularity case ($s < 1$).

This requires analysis techniques different from those used in [27]. Due to page limitation here, we will present the error analysis in a separate work.

5. Other FEMs for Darcy on Quadrilateral or Hybrid Meshes

5.1. **Mixed Finite Element Methods on Quadrilateral Meshes**

The primal pressure variable and its flux can be approximated simultaneously by rewriting the Darcy equation (1) as a system of first-order partial differential equations as follows [8]

$$
\begin{align*}
\mathbf{K}^{-1}\mathbf{u} + \nabla p &= 0, \quad \mathbf{x} \in \Omega, \\
\nabla \cdot \mathbf{u} &= f, \quad \mathbf{x} \in \Omega, \\
\mathbf{u} \cdot \mathbf{n} &= \mathbf{u}_N, \quad \mathbf{x} \in \Gamma^N, \\
p &= p_D, \quad \mathbf{x} \in \Gamma^D,
\end{align*}
$$

(22)

Mixed finite element methods satisfy the local mass conservation by design. The Piola transformation is utilized to enforce normal flux continuity for vector shape functions defined on quadrilaterals.
Define
\[ H_{N,0}(\text{div}, \Omega) = \{ v \in L^2(\Omega)^2 : \text{div}v \in L^2(\Omega), v|_{\Gamma^N} = 0 \}, \]
\[ H_{N,u_N}(\text{div}, \Omega) = \{ v \in L^2(\Omega)^2 : \text{div}v \in L^2(\Omega), v|_{\Gamma^N} = u_N \}. \]
The mixed variational formulation based on (22) is: Find \( u \in H_{N,u_N}(\text{div}, \Omega) \) and \( p \in L^2(\Omega) \) such that the following holds
\[
\begin{align*}
(K^{-1}u, v)_\Omega - (p, \nabla \cdot v)_\Omega &= -\langle p_D, v \cdot n \rangle_{\Gamma_D}, & \forall v \in H_{N,0}(\text{div}, \Omega), \\
-\langle \nabla \cdot u, q \rangle_{\Omega} &= -(f, q)_\Omega, & \forall q \in L^2(\Omega).
\end{align*}
\]
In this formulation, the Neumann condition is essential, whereas the Dirichlet condition becomes natural.

MFEMs are based on \( H(\text{div}) \)-conforming finite elements and the inf-sup condition [8]. Denote \( V = H(\text{div}; \Omega), W = L^2(\Omega) \). Let \( V_h \subset V \) and \( W_h \subset W \) be a pair of finite element spaces (for the flux and the primal variable respectively) satisfying the inf-sup condition. Furthermore, let \( U_h = V_h \cap H_{N,u_N}(\text{div}; \Omega) \) and \( V_h^0 = V_h \cap H_{N,0}(\text{div}; \Omega) \). A mixed finite element scheme can be formulated as: Find \( u_h \in U_h \) and \( p_h \in W_h \) such that
\[
\begin{align*}
\sum_{E \in \mathcal{E}_h} (K^{-1}u_h, v)_E - \sum_{E \in \mathcal{E}_h} (p_h, \nabla \cdot v)_E &= -\sum_{\gamma \in \Gamma_h^D} \langle p_D, v \cdot n \rangle_{\gamma}, & \forall v \in V_h^0, \\
-\sum_{E \in \mathcal{E}_h} \langle \nabla \cdot u_h, q \rangle_E &= -(f, q)_E, & \forall q \in W_h.
\end{align*}
\]
Let \( E = P_1P_2P_3P_4 \) be a convex quadrilateral with four vertices \( P_i(x_i, y_i), i = 1, 2, 3, 4 \) that are oriented counterclockwise. The bilinear mapping from the reference element \( \hat{E} = [0,1]^2 \) to the quadrilateral \( E \) can be expressed as
\[
\begin{align*}
x &= a_1 + a_2 \hat{x} + a_3 \hat{y} + a_4 \hat{x} \hat{y}, \\
y &= b_1 + b_2 \hat{x} + b_3 \hat{y} + b_4 \hat{x} \hat{y},
\end{align*}
\]
where \( \hat{x} = (\hat{x}, \hat{y}) \in \hat{E} \) and \( x = (x,y) \in E \). The eight coefficients \( a_i, b_i(i = 1, 2, 3, 4) \) can be directly calculated using the vertex coordinates. The Jacobian matrix of the bilinear mapping is
\[
J = \begin{bmatrix} a_2 + a_4 \hat{y} & a_3 + a_4 \hat{x} \\ b_2 + b_4 \hat{y} & b_3 + b_4 \hat{x} \end{bmatrix}.
\]
The Jacobian determinant is a linear polynomial in $\hat{x}, \hat{y}$:

$$J = \det(J) = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} + \begin{vmatrix} a_2 & a_4 \\ b_2 & b_4 \end{vmatrix} \hat{x} + \begin{vmatrix} a_4 & a_3 \\ b_4 & b_3 \end{vmatrix} \hat{y}. \quad (27)$$

By the Piola transformation, a vector function $\hat{\mathbf{v}}$ on $\hat{E}$ is transformed into a vector function on $E$ as

$$\mathbf{v}(\mathbf{x}) = \frac{\mathbf{J}(\hat{\mathbf{x}})}{J(\hat{\mathbf{x}})} \hat{\mathbf{v}}(\hat{\mathbf{x}}). \quad (28)$$

The Piola transformation maintains the normal fluxes for the vector basis functions on $E$ that are mapped from the vector basis functions on $\hat{E}$, and thus plays a fundamentally important role in constructing $H(\text{div})$ finite elements on quadrilaterals $[5, 8, 33]$. Moreover, it can be verified that ([8] p.140)

$$\nabla \cdot \mathbf{v} = \frac{1}{J(\hat{\mathbf{x}})} \hat{\nabla} \cdot \hat{\mathbf{v}}. \quad (29)$$

$H(\text{div})$-conforming finite elements on quadrilaterals based on the Piola transformation were investigated in [5] and necessary and sufficient conditions for optimal order approximations for vector fields and their divergences were established. The Raviart-Thomas element pair $(RT_{0}, Q_{0})$ on quadrilaterals was used to solve the Darcy or elliptic equations. First order convergence in velocity and pressure is obtained on general quadrilateral meshes provided the problem has full elliptic regularity on a convex domain. However, there may be no convergence for velocity divergence on general quadrilateral meshes. Our numerical results (Section 6 Example 2) indicate that optimal order convergence in velocity (and flux) can still be obtained in low regularity cases.

An advantage of the mixed finite elements constructed using the Piola transformation is the applicability to general quadrilateral meshes. However, an obvious restriction comes from solving indefinite linear systems or saddle-point problems. It is unclear whether the mixed element pair $(RT_{0}, Q_{0})$ for quadrilaterals can be combined with the mixed element pair $(RT_{0}, P_{0})$ for triangles to solve the Darcy (elliptic) equation on a hybrid mesh.

5.2. WGFEMs on Polygonal Meshes

In [27], a family of stabilized WG($P_{k+1}, P_{k}; P_{d}^{l}$) (dimension $d = 2, 3$, integer $k \geq 0$) finite element schemes have been developed for general polygonal
or polyhedral meshes. The schemes can be applied to distorted meshes that contain different types of polygons and polyhedra, e.g., a mix of triangles and quadrilaterals in two dimensions. The lowest-order WG finite element in this family in two dimensions is \((P_1, P_0; P_2^0)\). The interior basis functions are therefore one degree higher than the elements developed in this paper. Importantly, for the \(WG(P_1, P_0; P_2^0)\) elements in [27], the discrete weak gradients are specified in \(P_2^0\), whereas for the lowest-order WG elements in this paper, the discrete weak gradients are specified in \(RT_{[0]}\) or \(RT_0\).

The WG \((P_1, P_0; P_2^0)\) finite element scheme for the Darcy equation on a quadrilateral mesh is: Let \(E_h\) be a quadrilateral mesh and define

\[
\mathcal{B}_h(p_h, q) = \mathcal{A}_h(p_h, q) + \mathcal{S}_h(p_h, q),
\]

where \(\mathcal{A}_h(p_h, q)\) has the same form as the one shown in (13), but is defined for shape functions that are linear polynomials in element interiors and constants on edges. Here \(\mathcal{S}_h\) is a stabilization term,

\[
\mathcal{S}_h(p_h, q) = \rho \sum_{E \in E_h} \sum_{\gamma \subset E} h^{-1}\langle Q_h^0(p_h^\gamma) - p_h^\gamma, Q_h^0(q^\gamma) - q^\gamma \rangle_{\gamma}.
\]

and \(Q_h^0\) is the local \(L_2\)-projection into the space of constant functions on an edge.

As shown in [27], the above WG scheme with stabilization is locally mass-conservative and produces continuous normal fluxes. Furthermore, it has optimal order convergence in pressure. However, an optimal value for the stabilization parameter \(\rho\) is unclear, as demonstrated by the numerical results in Section 6 Example 2. Computation of a velocity field was not discussed in [27] and hence requires further investigation. One possible approach would be via a postprocessing procedure, applying an interpolation operator to the normal flux on the edges to obtain a numerical velocity that is in the global Raviart-Thomas space (a finite dimensional subspace of \(H(\text{div}, \Omega)\)).

6. Numerical Experiments

In this section, we test the lowest-order WG finite element method on quadrilateral meshes and hybrid meshes. We compare this new WG method with the \(WG(P_1, P_0; P_2^0)\) method in [27] and with the mixed finite element method in [5] based on the Piola transformation.
For quadrilateral mesh generation, see [29]. For requirements on quadrilateral mesh quality, see [16, 36]. For Matlab implementation techniques for related finite element methods, see [2, 6, 11, 25].

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{mesh_types.png}
\caption{Four types of meshes for numerical experiments}
\end{figure}

We shall consider these four types of meshes.

- **Type I: Logically rectangular meshes.** This family of meshes adopted from [33] is asymptotically parallelogram. Specifically, the quadrilateral mesh points are

  \[ \begin{align*}
  x &= \hat{x} + 0.06 \sin(2\pi \hat{x}) \sin(2\pi \hat{y}), \\
  y &= \hat{y} - 0.05 \sin(2\pi \hat{x}) \sin(2\pi \hat{y}),
  \end{align*} \]

  where \((\hat{x}, \hat{y})\) are the corresponding rectangular mesh points.

- **Type II: Asymptotically parallelogram trapezoidal meshes.** See [4, 5].

- **Type III: Quadrilaterals meshes obtained from a triangular mesh.** An initial quadrilateral mesh is obtained by refining a triangular mesh. Then finer meshes are obtained from regular refinement of quadrilateral meshes. This results in a family of asymptotically parallelogram quadrilateral meshes, as discussed in [4] (p.918).

- **Type IV: Hybrid meshes consisting of quadrilaterals and triangles.**

The quality of an initial Type III quadrilateral mesh may be poor, as is shown in Figure 2. But successive refinement improves mesh quality. More details about conditions and quality of quadrilateral meshes can be found in [12].

**Example 1 (Smooth solution).** First, we consider a simple example with \(\Omega = (0,1)^2, K = I_2\) (order 2 identity matrix), a known analytical
\[
\begin{array}{|c|c|c|c|c|c|}
\hline
1/h & \|p - p_h\| & \|u - u_h\| & \|(u - u_h) \cdot n\| & \min(p_h) & \max(p_h) \\
\hline
\text{Type I meshes} & & & & & \\
16 & 4.2299e-2 & 1.3818e-1 & 1.9030e-1 & 7.3817e-3 & 9.8536e-1 \\
64 & 1.0604e-2 & 3.4320e-2 & 4.7204e-2 & 5.5932e-4 & 9.9897e-1 \\
\text{Conv.rate} & 0.994 & 1.010 & 1.015 & N/A & N/A \\
\hline
\text{Type II meshes} & & & & & \\
16 & 3.8217e-2 & 3.7097e-1 & 4.9134e-1 & 7.1831e-3 & 9.6817e-1 \\
32 & 1.8595e-2 & 1.6574e-1 & 2.2054e-1 & 2.0653e-3 & 9.9358e-1 \\
64 & 9.2692e-3 & 7.5206e-2 & 1.0084e-1 & 5.556e-4 & 9.9883e-1 \\
\text{Conv.rate} & 1.058 & 1.108 & 1.097 & N/A & N/A \\
\hline
\text{Type IV meshes} & & & & & \\
8 & 7.2665e-2 & 2.7675e-1 & 3.3213e-1 & 1.3150e-2 & 9.6983e-1 \\
16 & 3.6563e-2 & 1.3839e-1 & 1.6405e-1 & 3.3015e-3 & 9.9233e-1 \\
\text{Conv.rate} & 0.995 & 0.999 & 1.008 & N/A & N/A \\
\hline
\end{array}
\]

Table 1: Example 1 (smooth solution): WG\((Q_0, Q_0; RT_{01})\) results for three types of meshes

Figure 3: Example 1 (smooth solution): Profiles of numerical pressure and velocity from the lowest-order weak Galerkin method on a quadrilateral mesh (left) and a hybrid mesh (right). Both have mesh size \(h = 1/16\).
solution for pressure $p(x, y) = \sin(\pi x) \sin(\pi y)$ and a homogeneous Dirichlet boundary condition on the entire boundary. It is easy to see that $f(x, y) = 2\pi^2 \sin(\pi x) \sin(\pi y)$. We test this example on three different types meshes (Type I, III, IV). The pressure solution is infinitely smooth, so the approximation accuracy is dictated by mesh quality. The convergence in pressure, velocity, flux shown in Table 1 is first order as expected.

**Example 2 (Low regularity solution).** For this example, $\Omega = (0, 1)^2$, $K = I_2$, a homogeneous Dirichlet boundary condition is posed on the entire boundary, and an analytical solution for pressure is specified as

$$p(x, y) = x(1-x)y(1-y)\left(\sqrt{x^2+y^2}\right)^{-(2-a)},$$

where $a \in (0, 1]$ is a regularity parameter. The right hand side function in the PDE is derived accordingly. The pressure admits a corner singularity at the origin. The pressure solution is in Sobolev space $p \in W^{2,r}(\Omega)$ with $r = 2/(2-a) + \delta$ for any $\delta > 0$ [35]. By the Sobolev embedding theorem, it is known [24] that

$$p \in H^1_0(\Omega) \cap H^{1+a-\delta}(\Omega), \quad p \notin H^{1+a}(\Omega),$$

for any small $\delta > 0$. A more accurate characterization based on Besov spaces can be found in [24]. The regularity order is close to $1 + a$. It is also interesting to note that the pressure solution has a minimum 0 and a maximum $\sqrt{2^a + a^a}/(2 + a)^{2+a}$ (attained at point $(\frac{a}{2+a}, \frac{a}{2+a})$).

For $a = 0.4$, the exact pressure maximum is 0.194783879417743. We test the WG($Q_0, Q_0; RT_0$) method in this paper, the WG($P_1, P_0; P^2_0$) method with stabilization in [27], and the mixed method ($RT_0, Q_0$) (based on the Piola transformation) in [5] on a family of asymptotically parallelogram trapezoidal meshes used in [5].

Table 2 provides results obtained from using WG ($Q_0, Q_0; RT_0$) on Type II (asymptotically parallelogram trapezoidal) meshes. It is observed that the pressure error exhibits about 1st order convergence. The errors in velocity and flux exhibit slightly better than order 0.4 convergence. Overshoots for pressure maximum are observed.

Table 3 provides the numerical results obtained from using the WG ($P_1, P_0; P^2_0$) method studied in [27] on the same trapezoidal meshes. It can be observed that the convergence rate in pressure error is about order 1.4.
\[
\|p - p_h\| \quad \|u - u_h\| \quad \|(u - u_h) \cdot n\| \quad \min(p_h) \quad \max(p_h)
\]

<table>
<thead>
<tr>
<th>(1/h)</th>
<th>(|p - p_h|)</th>
<th>(|u - u_h|)</th>
<th>(|(u - u_h) \cdot n|)</th>
<th>(\min(p_h))</th>
<th>(\max(p_h))</th>
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<tr>
<td>8</td>
<td>2.2025e-02</td>
<td>3.7831e-01</td>
<td>1.1725e+00</td>
<td>2.6101e-03</td>
<td>2.3095e+01</td>
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<td>1.0418e-02</td>
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<td>8.5877e-01</td>
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<tr>
<td>64</td>
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<td>1.5788e-01</td>
<td>4.8109e-01</td>
<td>4.2083e-05</td>
<td>1.9489e+01</td>
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<tr>
<td>128</td>
<td>1.2805e-03</td>
<td>1.1938e-01</td>
<td>3.6315e-01</td>
<td>1.0350e-05</td>
<td>1.9479e+01</td>
</tr>
<tr>
<td>Conv.rate</td>
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<td>0.416</td>
<td>0.422</td>
<td>N/A</td>
<td>N/A</td>
</tr>
</tbody>
</table>

Table 2: Example 2 (low regularity): Results of the lowest order WG\((Q_0, Q_0; RT_0)\) method on asymptotically parallelogram trapezoidal meshes (Type II)

<table>
<thead>
<tr>
<th>(1/h)</th>
<th>(|p - p_h|)</th>
<th>(\min(p_h))</th>
<th>(\max(p_h))</th>
<th>(|p - p_h|)</th>
<th>(\min(p_h))</th>
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<td>(\rho = 1)</td>
<td>9.950E-2</td>
<td>7.848E-1</td>
<td>9.979E-3</td>
<td>2.775E-3</td>
<td>7.451E3</td>
<td>1.733E1</td>
</tr>
<tr>
<td>8</td>
<td>3.586E-2</td>
<td>5.651E-1</td>
<td>3.745E-3</td>
<td>7.575E-2</td>
<td>4.146E-1</td>
<td>1.892E-1</td>
</tr>
<tr>
<td>16</td>
<td>1.320E-2</td>
<td>4.165E-1</td>
<td>1.414E-3</td>
<td>1.715E-2</td>
<td>1.028E-1</td>
<td>1.938E-1</td>
</tr>
<tr>
<td>32</td>
<td>4.925E-3</td>
<td>3.110E-1</td>
<td>5.357E-4</td>
<td>4.140E-3</td>
<td>1.030E-1</td>
<td>1.945E-1</td>
</tr>
<tr>
<td>64</td>
<td>1.850E-3</td>
<td>2.340E-1</td>
<td>2.031E-4</td>
<td>1.023E-3</td>
<td>1.947E-1</td>
<td></td>
</tr>
<tr>
<td>128</td>
<td>7.450E-4</td>
<td>1.1897E-1</td>
<td>1.0298E-5</td>
<td>1.023E-4</td>
<td>1.9478E-1</td>
<td></td>
</tr>
<tr>
<td>Conv.rate</td>
<td>1.437</td>
<td>N/A</td>
<td>N/A</td>
<td>1.404</td>
<td>N/A</td>
<td>N/A</td>
</tr>
</tbody>
</table>

Table 3: Example 2 (low regularity): Results of WG\((P_1, P_0; P_0^2)\) with stabilization [27] on asymptotically parallelogram trapezoidal meshes (Type II)

<table>
<thead>
<tr>
<th>(1/h)</th>
<th>(|p - p_h|_{L^2})</th>
<th>(|u - u_h|)</th>
<th>(\min(p_h))</th>
<th>(\max(p_h))</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>2.1071e-02</td>
<td>3.6181e-01</td>
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<td>2.2682e-01</td>
</tr>
<tr>
<td>16</td>
<td>1.0243e-02</td>
<td>2.7261e-01</td>
<td>7.0278e-04</td>
<td>2.0042e-01</td>
</tr>
<tr>
<td>32</td>
<td>5.1021e-03</td>
<td>4.165E-1</td>
<td>1.7034e-04</td>
<td>1.9528e-01</td>
</tr>
<tr>
<td>64</td>
<td>2.5549e-03</td>
<td>4.140E-1</td>
<td>4.172E-5</td>
<td>1.945E-1</td>
</tr>
<tr>
<td>128</td>
<td>1.2799e-03</td>
<td>2.340E-1</td>
<td>1.030E-5</td>
<td>1.947E-1</td>
</tr>
<tr>
<td>Conv.rate</td>
<td>1.010</td>
<td>0.401</td>
<td>N/A</td>
<td>N/A</td>
</tr>
</tbody>
</table>

Table 4: Example 2 (low regularity): Results of MFEM based on the Piola transformation [5] on asymptotically parallelogram trapezoidal meshes (Type II)

for different stabilization parameter values \(\rho = 1\) and \(\rho = 10\). However, the maximum of the numerical pressure clearly depends on the choice of the stabilization parameter \(\rho\). For \(\rho = 1\), the method shows the expected convergence in pressure errors, but a fairly large overshoot is still observed for a fine mesh with \(h = 1/128\). Increasing stabilization parameter value results
in increase of the condition number of the discrete linear system. Further investigation is needed to determine the optimal stabilization parameter and to compute element-wise velocity for this WG method.

Table 4 provides numerical results obtained from using the mixed finite element method in [5] on the same trapezoidal meshes. It can be observed that the pressure error exhibits 1st order convergence and the velocity error exhibits order 0.4 convergence. Overshoots in pressure maximum are also observed.

**Example 3 (Quadrilateral meshes in polar coordinates).** In this example, we solve the Darcy equation on a circular sector domain that is described in polar coordinates

\[
\Omega = \{(r, \theta) : r_a \leq r \leq r_b, 0 \leq \theta \leq \frac{\pi}{2}\},
\]

where \(0 < r_a < r_b\) are respectively the inner and outer radii of an annulus. For simplicity, we assume the permeability matrix \(K\) has principal axes in the radial and theta directions, i.e.,

\[
K = \begin{bmatrix}
\cos(\theta) & -\sin(\theta) \\
\sin(\theta) & \cos(\theta)
\end{bmatrix}
\begin{bmatrix}
\lambda & 0 \\
0 & \mu
\end{bmatrix}
\begin{bmatrix}
\cos(\theta) & \sin(\theta) \\
-\sin(\theta) & \cos(\theta)
\end{bmatrix},
\]

or in Cartesian coordinates,

\[
K = \begin{bmatrix}
\frac{\lambda x^2 + \mu y^2}{x^2 + y^2} & (\lambda - \mu) \frac{xy}{x^2 + y^2} \\
(\lambda - \mu) \frac{xy}{x^2 + y^2} & \frac{\lambda y^2 + \mu x^2}{x^2 + y^2}
\end{bmatrix}.
\]

While the permeability matrix have the same form in both polar and Cartesian coordinates when \(\lambda = \mu\), many applications exhibit strong anisotropy. For example in the biological problems described in [26, 19], the radial permeability \(\lambda\) greatly exceeds the angular permeability \(\mu\).

The gradient operators in the Cartesian and polar coordinates are related as

\[
\begin{bmatrix}
\frac{\partial}{\partial x} \\
\frac{\partial}{\partial y}
\end{bmatrix} p = \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & \frac{1}{r}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial}{\partial r} \\
\frac{\partial}{\partial \theta}
\end{bmatrix} p.
\]

Accordingly, the Darcy velocity is

\[
v = -K \begin{bmatrix}
\frac{\partial}{\partial x} \\
\frac{\partial}{\partial y}
\end{bmatrix} p = \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}
\begin{bmatrix}
\lambda \frac{\partial}{\partial r} \\
\mu \frac{1}{r} \frac{\partial}{\partial \theta}
\end{bmatrix} p,
\]

19
and the source term is
\[ f = \left[ \frac{\partial_x}{\partial y} \right] \cdot \mathbf{v} = \left[ \begin{array}{cc} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{array} \right] \left[ \begin{array}{c} \frac{\partial_r}{\frac{1}{r} \partial_\theta} \end{array} \right] \cdot \mathbf{v}. \]

We consider an exact pressure solution
\[ p(x, y) = xy, \]
and calculate the right-hand side in the Darcy equation as
\[ f(x, y) = -4(\lambda - \mu) \frac{xy}{x^2 + y^2}. \]

All four sides of the sector domain are posed as Dirichlet boundary conditions with data prescribed by the exact pressure solution.

For testing anisotropy in the radial and angular directions, we fix \( \lambda = 1 \) and vary \( \mu \) from 1 down to \( 10^{-3} \) or \( 10^{-6} \). Our numerical results demonstrate good approximation for pressure, velocity, and flux in the cases of highly anisotropic permeability fields.

For numerical tests, we set \( n_r \) as the number of uniform partitions in the radial direction and \( n_\theta = 2n_r \) as the number of uniform partitions in the angular direction.

Numerical results applying WG \( (Q_0, Q_0; RT_0) \) to this example with \( \lambda = 1 \) and \( \mu = 10^{-3} \) (strong anisotropy) are provided in Table 5. Nearly first
order convergence can be observed in pressure and velocity errors and in the edgewise bulk flux. No overshoots or undershoots in pressure are observed.

Figure 4 shows profiles of numerical pressure and velocity for $n_r = 8$, $n_\theta = 16$, with fixed $\lambda = 1$ but two different $\mu$ values (1 and 0.1). As the ratio $\lambda/\mu$ increases, the radial dominance in numerical velocity becomes obvious.

This type of problem could be solved by isogeometric finite element methods [19], an approach that requires different data structures. Here we demonstrate how the novel weak Galerkin FEMs could be used in a simple way. One could also use mixed finite element methods [5, 21] or multi-point flux mixed finite element methods [33] on quadrilaterals, but the Piola transformation needs to be used. The lowest order WG finite element method is easier by adopting the non-mapped approach and allows hybrid meshes comprised of quadrilaterals and triangles.

**Example 4 (Highly heterogeneous permeability fields).** This example originates from [14] and has been tested by many researchers and also in [23] on rectangular and triangular meshes. Here we extend this example to a trapezoidal domain

$$
\Omega_2 = \{(x, y) : 0 \leq y \leq \frac{x + 1}{2}, 0 \leq x \leq 1\},
$$

and a circular sector domain

$$
\Omega_3 = \{(r, \theta) : 2 \leq r \leq 3, \frac{\pi}{6} \leq \theta \leq \frac{\pi}{3}\}.
$$

For the unit square domain $\Omega_1 = [0, 1]^2$, as discussed in [23], Dirichlet boundary conditions are specified on the left and right sides, whereas Neumann boundary conditions are specified on the top and bottom sides:

$$
p = 1, \text{ left; } p = 0, \text{ right; } -(K \nabla p) \cdot n = 0, \text{ top or bottom.}
$$
Figure 5: Example 4 (heterogeneous permeability fields): Pressure and velocity profiles.
(a) Permeability profile on $\Omega_1 = [0, 1]^2$; (b) Rectangular domain: WG($Q_0, Q_0; RT_{[0]}$) on a uniform rectangular mesh with $h = 1/40$; (c) Trapezoidal domain: WG ($Q_0, Q_0; RT_{[0]}$) on a quadrilateral with $h = 1/40$; (d) Circular sector domain: WG ($Q_0, Q_0; RT_{[0]}$) on a quadrilateral with $h = 1/40$. Flow patterns differ slightly.

The trapezoidal domain $\Omega_2$ is mapped from the unit square with corresponding domain sides and boundary conditions. The circular sector domain $\Omega_3$ is also a mapping image of the unit square (left side $\rightarrow$ inner circle, right side $\rightarrow$ outer circle, etc.)

The new WG method for quadrilateral meshes is tested on all three domains. Figure 5 shows that flow patterns differ slightly. For the circular sector domain, flow paths in the radial direction are more pronounced. But no overshoots or undershoots in numerical pressure are observed.
7. Concluding Remarks

We have developed an easily implemented and practically useful Darcy solver that can be used on quadrilateral meshes, triangular meshes, and hybrid meshes consisting of quadrilaterals and triangles. This Darcy solver is developed utilizing the novel ideas of weak Galerkin finite elements. The pressure unknowns are constants inside (triangular and quadrilateral) element interiors and on edges. The discrete weak gradients of these constant basis functions are computed inexpensively. On triangles and rectangles they are obtained by direct calculations, and on general quadrilaterals they are obtained by solving $4 \times 4$ symmetric positive-definite linear systems. These discrete weak gradients are used to approximate the classical gradient in the variational formulation. The finite element scheme results in a sparse symmetric positive-definite discrete linear system that can be solved by standard linear solvers. After the numerical pressure $p_h$ is solved, a local $L^2$-projection of $-K \nabla w, d p_h$ is conducted to obtain the numerical velocity. (The $L^2$-projection is not required if $K$ is an element-wise constant scalar matrix.) The normal flux can then be computed in a straightforward manner.

This new approach has some obvious nice features, including

1. Local mass conservation and continuity of normal flux
2. Lowest-order approximants (constants inside elements and on edges)
3. Unified treatment of triangles, rectangles, and quadrilaterals
4. No necessity for stabilization
5. Sparse SPD global linear systems.

The limitation of the method is the “asymptotically parallelogram” assumption on quadrilaterals, although this assumption is widely adopted. This is not a serious limitation, since any polygonal domain can be partitioned into a family of asymptotically parallelogram, shape-regular quadrilateral meshes [4] (p. 918).

Different than the existing work in [3, 5], convergence in divergence is not a main concern of this paper. The main theme of this paper is about development of an easy-to-use Darcy solver on quadrilateral meshes. The focus is on local conservation, normal flux continuity, and convergence in pressure, velocity, and flux. Our results indicate that the unmapped approach (not using the Piola transformation) combined with the novel weak Galerkin methodology can do a good job in this regard.
An extension of this lowest order weak Galerkin finite element method to three-dimensional domains is not difficult but requires some attention to details [18]. First, the faces of a hexahedron may not be planar. Integration on a face or interior of the hexahedron requires mapping back to the unit square or the unit cube and involves higher computational costs. By contrast, an integral on a quadrilateral can be simply split into integrals on two triangles. More delicate handling is needed for fluxes on non-planar faces.

Acknowledgements: The first was partially supported by US National Science Foundation under grant DMS-1419077. The third author was partially supported by US National Science Foundation under grant DMS-1419077 and a graduate fellowship from the Center of Interdisciplinary Mathematics and Statistics at Colorado State University.

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