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A Comparative Study of Locally Conservative Numerical Methods for Darcy’s Flows

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Abstract

This paper presents a comparative study on locally mass-conservative numerical methods for Darcy’s flows. The classical mixed finite element method (MFEM) is compared with the newly developed discontinuous finite volume method (DFVM) with and without weak over-penalization (WOP). These numerical methods are tested on three representative problems in porous media flows. In particular, locality, accuracy of numerical solutions, computational costs, and implementation issues are examined. The study indicates that the discontinuous finite volume methods could be viable alternatives to the classical mixed finite element method for Darcy’s flows.

Keywords: Darcy’s flows, discontinuous finite volume methods, locally mass-conservative, mixed finite element methods, weak over-penalization

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1. Introduction

The Darcy’s law plays a fundamental role in porous media flows [7, 10, 11, 20]. Numerical methods for the Darcy’s law have to be locally mass-conservative to ensure correctness and usefulness of the obtained numerical velocities in subsequent transport simulations [16]. The numerical experiments in [16] have demonstrated that violation of local mass conservation results in severe overshoots and/or undershoots and loss of accuracy of concentrations in follow-up transport simulations.

It is well known that the continuous Galerkin (CG) method is not locally mass-conservative. Postprocessing techniques have been developed [9, 18] to compute locally conservative fluxes from CG solutions. While these techniques could be used to salvage the legacy codes developed in the early days, direct locally mass-conservative numerical methods are preferred, since they shall offer more convenience for practical computations.

The mixed finite element methods [5, 6, 14, 15], the node-centered finite volume methods (FVM), and the discontinuous Galerkin (DG) finite element methods are all locally conservative by design. Recently the discontinuous finite volume method [22], DFVM with weak over-penalization [13], and the enriched Galerkin (EG) method [16] have been developed. These new methods are also locally conservative.
In this paper, we conduct a comparative study of these locally conservative numerical methods for Darcy’s flows. In particular, the mixed finite element method and the discontinuous finite volume method with and without over-penalization are tested on three representative flow problems in porous media. Their locality, accuracy of numerical solutions, computational costs, and implementation easiness are compared.

The Darcy’s law is usually formulated as

\[
\begin{align*}
-\nabla \cdot (K \nabla p) & \equiv \nabla \cdot \mathbf{u} = f, \quad \mathbf{x} \in \Omega, \\
-K \nabla p \cdot \mathbf{n} & = u_N, \quad \mathbf{x} \in \Gamma_N, \\
p & = p_D, \quad \mathbf{x} \in \Gamma_D,
\end{align*}
\]

(1)

where \( \Omega \subset \mathbb{R}^d (d = 2, 3) \) is a bounded polygonal or polyhedral domain, \( p \) the unknown pressure, \( K \) a permeability tensor that is uniformly symmetric positive-definite, \( f \) a source term, \( p_D, u_N \) are respectively Dirichlet and Neumann boundary data, \( \mathbf{n} \) the unit outward normal on\( \Gamma = \partial \Omega \), which has a nonoverlapping decomposition \( \Gamma = \Gamma_D \cup \Gamma_N \). It is assumed that \( \Gamma_D \neq \emptyset \), so that the problem has a unique solution. In numerical simulations, it is conventional to assume that \( K \) is piecewise constant on a given mesh.

2. Locally Conservative Numerical Methods

In this section, we present three locally mass-conservative numerical methods for the Darcy’s equation (1): the discontinuous finite volume method, the discontinuous finite volume method with weak over-penalization, and the classical mixed finite element method. The features of the resulting discrete linear systems, condition numbers, and error estimates of these methods will be briefly discussed.

Throughout the paper, we adopt the conventional notations to use \( L_2(\Omega) \) to denote the space of the Lebesgue square integrable functions on \( \Omega \); \( H^k(\Omega) (k = 1, 2) \) denotes the subspace of \( L_2(\Omega) \) functions whose weak derivatives up to \( k \)-th order are also square integrable. Accordingly, \( \| \cdot \|_{L_2(\Omega)}, \| \cdot \|_{H^1(\Omega)}, \| \cdot \|_{H^2(\Omega)} \) denote the norms in the corresponding spaces.

2.1. Discontinuous Finite Volume Method

Let \( \mathcal{T}_h \) be a quasi-uniform conforming (no hanging nodes) triangular mesh on \( \Omega \) and \( \mathcal{T}_h^* \) be the dual partition defined in [22], see Figure 1. Let \( \mathcal{E}_h \) denote the set of all edges, \( \mathcal{E}_h^I \) the set of all interior edges, \( \mathcal{E}_h^D \) the set of the edges on \( \Gamma_D \), and \( \mathcal{E}_h^N \) the set of the edges on \( \Gamma_N \).

We define respectively a finite dimensional space for the piecewise linear trial functions and a finite dimensional space for the piecewise constant test functions as follows

\[
P_h = \{ p \in L_2(\Omega) : p|_T \in P_1(T), \ \forall T \in \mathcal{T}_h \},
\]

(2)
Theorem 1. Let $p$ and $p_h$ be respectively the solution of (1) and (7). The following hold
\begin{align}
|p - p_h| & \leq Ch\|p\|_{H^1(\Omega)}, \quad (11) \\
\|p - p_h\|_{L^2(\Omega)} & \leq Ch^2\|p\|_{H^1(\Omega)}. \quad (12)
\end{align}

For DfvmP1, the global coefficient matrix is symmetric if the permeability is a constant tensor on each element. Its condition number is generally $O(h^{-2})$ and also relies on the penalty factor. We refer to [19] for a detailed discussion on this type of dependence.

After a numerical pressure $p_h$ is obtained by solving (7), a numerical velocity $u_h$ is then computed as follows [16]
\begin{align}
\mathbf{u}_h &= -\nabla p_h, \quad \mathbf{x} \in T, \quad T \in \mathcal{T}_h, \\
\mathbf{u}_h \cdot \mathbf{n} &= -[(\nabla p_h \cdot \mathbf{n}) + \frac{\alpha_e}{h_e} (p_h|_{T_1} - p_h|_{T_2})], \quad \mathbf{x} \in e = \partial T_1 \cap \partial T_2, \quad T_1, T_2 \in \mathcal{T}_h \quad \text{and} \quad \mathbf{n} \text{ exterior to } T_1, \\
\mathbf{u}_h \cdot \mathbf{n} &= u_N, \quad \mathbf{x} \in \Gamma_N, \\
\mathbf{u}_h \cdot \mathbf{n} &= -\nabla p_h \cdot \mathbf{n} + \frac{\alpha_e}{h_e} (p_h - p_D), \quad \mathbf{x} \in \Gamma_D.
\end{align}
2.2. Discontinuous Finite Volume Method with Weak Over-penalization

An interesting variant of DfvmP1 comes with the introduction of weak over-penalization and abandon of the average/jump terms of the trial/test functions [13]. Our discontinuous finite volume method with linear shape functions and weak over-penalization (DfvmP1WOP) reads as: Seek \(p_h \in P_h\) so that

\[
\mathcal{A}_h(p_h, q_h) = \mathcal{T}_h(q_h), \quad \forall q_h \in P_h,
\]

where the bilinear and linear forms are defined for any \(p, q \in V_h\) as

\[
\mathcal{A}_h(p, q) = -\sum_{T \in T_h} \sum_{i=1}^3 \int_{P_{x_i}} (K \nabla p \cdot n)(I^*_h q) ds + \sum_{e \in E^*_h} h_e^{-1} [I^*_h p][I^*_h q],
\]

\[
\mathcal{T}_h(q) = \sum_{T \in T_h} (f, I^*_h q)_T + \sum_{e \in E^*_h} h_e^{-2} (I^*_h q_D)(I^*_h q) + \sum_{e \in E^*_h} (u_N, q)_e.
\]

For DfvmP1WOP, the procedure for computing a numerical velocity based on a numerical pressure is the same as that for DfvmP1. Error estimates similar to (11) have been established in [13].

**Theorem 2.** Let \(p\) and \(p_h\) be respectively the solution of (1) and (14). If \(p \in H^2(\Omega)\), then

\[
\|p - p_h\|_H \leq C h \|p\|_{H^2(\Omega)},
\]

\[
\|p - p_h\|_{L^2(\Omega)} \leq C h^2 \|p\|_{H^2(\Omega)} + \|f\|_{H^1(\Omega)},
\]

where the mesh-dependent energy norm for \(q \in H^1(\Omega) + P_h\) is defined as

\[
\|q\|_h^2 = \sum_{T \in T_h} \|\nabla q\|_{L^2(T)}^2 + \sum_{e \in E_h} h_e^{-2} [I^*_h q]_e^2.
\]

A noticeable feature of DfvmP1WOP is its easiness in implementation. On each triangular element, one can choose the three edge-midpoint-oriented Lagrangian linear polynomials as a set of local basis functions. The global coefficient matrix has a clear block structure. It is symmetric if the permeability tensor is a constant on each element. Its condition number behaves like \(O(h^{-3})\). However, a simple block diagonal preconditioner can be constructed utilizing the aforementioned basis functions [4]. In particular, the preconditioner contains the global mass matrix and a global jump matrix [13]. The global jump matrix has respectively \(2 \times 2\) or \(1 \times 1\) blocks

\[
\frac{1}{|e|^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ |e|^2 \end{bmatrix},
\]

for an interior or Dirichlet boundary edge with \(|e|\) being the edge length. Note that Neumann boundary edges don’t contribute to the global jump matrix.

2.3. Mixed Finite Element Method

The Darcy’s law (1) can be rewritten as a system of first order partial differential equations about the pressure and velocity as follows [6, 7]

\[
\begin{cases}
K^{-1} u + \nabla p = 0, & x \in \Omega, \\
\nabla \cdot u = f, & x \in \Omega, \\
u \cdot n = u_N, & x \in \Gamma_N, \\
p = p_D, & x \in \Gamma_D.
\end{cases}
\]

Let \(T_h\) be as before. We define two finite element spaces \(V_h\) and \(Q_h\) respectively for velocity and pressure as follows

\[
V_h = \{ v : v \in L^2(\Omega)^2, \nabla \cdot v \in L^2(\Omega), v|_T \in RT_1(T), v \cdot n|_{\Gamma_N} = 0 \},
\]

\[
Q_h = \{ q : q \in H^1(\Omega) + P_h \}.
\]
\[ Q_h = \{ q \in L^2(\Omega) : q|_T = P_k(T) \}, \quad (23) \]

where RT\_k is the k-th order Raviart-Thomas element [6].

Let \( u_h = u_0 + \bar{u} \), where \( \bar{u} \) is a known function such that \( \bar{u} \cdot n = u_N \) on \( \Gamma_N \). Then the mixed finite element method (MFEMRTk) for problem (21) seeks \((u_0, p_h) \in V_h \times Q_h\) such that for any \( v \in V_h \) and \( q \in Q_h\)

\[ A_h(u_0, v) - B_h(v, p_h) = -(p_D, v \cdot n)_{\Gamma_D} - A_h(\bar{u}, v), \quad (24) \]

\[ B_h(u_0, q) = (f, q)_\Omega - B_h(\bar{u}, q), \quad (25) \]

where

\[ A_h(u_0, v) = \int_\Omega (K^{-1}u_0) \cdot vdx, \quad (26) \]

\[ B_h(v, q) = \int_\Omega (\nabla \cdot v)qdx. \quad (27) \]

**Theorem 3.** Let \((u, p)\) and \((u_h, p_h)\) be respectively the solution of (21) and (24)-(25) and \( s \leq k + 1 \). Then [6]

\[ \|u - u_h\|_{L^2(\Omega)} \leq Ch^s \|u\|_{H^s(\Omega)}, \quad (28) \]

\[ \|p - p_h\|_{L^2(\Omega)} \leq Ch^s (\|u\|_{H^s(\Omega)} + \|p\|_{H^s(\Omega)}). \quad (29) \]

Note that MFEMRT0 has only first order accuracy, since the shape functions are not complete degree-one polynomials. MFEMRT1 does have first order accuracy but involves partial quadratic polynomials. An advantage of MFEM is that a numerical velocity is obtained directly without postprocessing like the other two methods, but more unknowns have to be solved, as shown in Table 1. Another feature of MFEM is the saddle-point problem [2], which is more difficult to solve than the definite linear systems obtained from the other two methods.

However, we would like to point out that when the mixed finite element method is applied on rectangular grids, an appropriate quadrature rule can be chosen to reduce the saddle problem to a symmetric positive definite cell-centered finite volume formulation. Similarly, on general hexahedral grids, a quadrature rule can be chosen such that the problem is equivalent to a multipoint flux approximation method. In both cases, the number of degrees of freedom can be reduced without sacrificing accuracy, see [21] and the references therein.
Table 1: Comparison of locality and degrees of freedom (DOFs) for the three locally conservative numerical methods on $n \times n \times 2$ structured triangular meshes

<table>
<thead>
<tr>
<th></th>
<th>DfvmP1</th>
<th>DfvmP1WOP</th>
<th>MfemRT0</th>
<th>MfemRT1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Locality</td>
<td>subelements</td>
<td>subelements</td>
<td>elements</td>
<td>elements</td>
</tr>
<tr>
<td>DOFs</td>
<td>$6n^2$</td>
<td>$6n^2$</td>
<td>$5n^2$</td>
<td>$16n^2$</td>
</tr>
</tbody>
</table>

3. Numerical Experiments

In this section, we conduct numerical experiments to compare these three locally conservative numerical methods on three porous media flow problems with different permeability profiles. For all three test problems, we solve the Darcy’s equation (1) on the unit square $\Omega = (0,1)^2$ with the following boundary conditions

$$
p = 1, \text{ left; } \quad p = 0, \text{ right; } \quad u \cdot n = 0, \text{ elsewhere.}
$$

For simplicity of implementing DfvmP1, we set the penalty factor $\alpha_e = 1$ for all edges. In Figures 3, 4, 5, we plot respectively a numerical velocity obtained from one of these three methods on a $40 \times 40 \times 2$ triangular mesh, since the differences to the numerical velocities obtained from the other two methods are indistinguishable. For better visual effects, we magnify all numerical velocities by a factor of two.

Example 1: A thin channel. In this test problem, a thin channel $\Omega_c = [\frac{3}{10}, \frac{7}{10}] \times [\frac{4}{10}, \frac{5}{10}]$ is contained in the domain $\Omega = (0,1)^2$. The permeability is 0.1 on $\Omega_c$ and 0.001 on $\Omega \setminus \Omega_c$, see Figure 3. Clearly, the flow runs faster in the thin channel. No exact solution is known inside the domain for Example 1 (and Examples 2, 3), but the exact solution values are known from the given Dirichlet boundary conditions on the left and right sides. The errors of the numerical pressure at the nodes on these pieces of boundary can be measured and could be viewed as representative, see Table 2.

![Figure 3: Example 1. (a) A thin channel permeability profile; (b) The numerical pressure and velocity obtained by DfvmP1WOP on a 40x40x2 triangular mesh.](image)
Table 2: Example 1. Errors of numerical pressures at nodes on the left and right boundaries

<table>
<thead>
<tr>
<th></th>
<th>Left</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th>Error order</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>n=10</td>
<td>n=20</td>
<td>n=40</td>
<td>n=80</td>
<td></td>
</tr>
<tr>
<td>DfvmP1</td>
<td>6.774E-4</td>
<td>1.858E-4</td>
<td>4.873E-5</td>
<td>1.238E-5</td>
<td></td>
<td>$O(h^2)$</td>
</tr>
<tr>
<td>DfvmP1WOP</td>
<td>6.403E-4</td>
<td>1.760E-4</td>
<td>4.593E-5</td>
<td>1.168E-5</td>
<td></td>
<td>$O(h^2)$</td>
</tr>
<tr>
<td>MfemRT0</td>
<td>5.826E-2</td>
<td>2.979E-2</td>
<td>1.506E-2</td>
<td>7.566E-3</td>
<td></td>
<td>$O(h)$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Right</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th>Error order</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>n=10</td>
<td>n=20</td>
<td>n=40</td>
<td>n=80</td>
<td></td>
</tr>
<tr>
<td>DfvmP1</td>
<td>6.774E-4</td>
<td>1.858E-4</td>
<td>4.873E-5</td>
<td>1.238E-5</td>
<td></td>
<td>$O(h^2)$</td>
</tr>
<tr>
<td>DfvmP1WOP</td>
<td>6.437E-4</td>
<td>1.764E-4</td>
<td>4.598E-5</td>
<td>1.169E-5</td>
<td></td>
<td>$O(h^2)$</td>
</tr>
<tr>
<td>MfemRT0</td>
<td>5.824E-2</td>
<td>2.980E-2</td>
<td>1.506E-2</td>
<td>7.567E-3</td>
<td></td>
<td>$O(h)$</td>
</tr>
</tbody>
</table>

Example 2: A poorly permeable region. In Example 2, the domain $\Omega = (0, 1)^2$ contains a central subdomain $\Omega_c = \left[\frac{3}{8}, \frac{5}{8}\right] \times \left[\frac{1}{4}, \frac{3}{4}\right]$. The permeability is a diagonal tensor $K = KI_2$ with $K$ being $10^{-3}$ on $\Omega_c$ and $10^{-1}$ elsewhere, as illustrated in Figure 4. It can be clearly observed that the flow takes detour due to the low permeability in the central region. Table 3 tabulates the condition numbers of the three locally conservative numerical methods on Example 2 with $n \times n \times 2$ triangular meshes. For DfvmP1WOP, the condition number growth is 4th order, due to the weak over-penalization. The condition number growth rate of MfemRT0 is not so clear, since it is a saddle-point problem.

Table 3: Condition numbers of three locally conservative numerical methods on Example 2 with $n \times n \times 2$ triangular meshes

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th>General</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>n=8</td>
<td>n=16</td>
<td>n=32</td>
<td>n=64</td>
<td></td>
<td></td>
</tr>
<tr>
<td>DfvmP1</td>
<td>8.45E3</td>
<td>3.51E4</td>
<td>1.42E5</td>
<td>5.69E5</td>
<td>$O(n^r)$</td>
<td></td>
</tr>
<tr>
<td>DfvmP1WOP</td>
<td>3.87E5</td>
<td>6.43E6</td>
<td>1.03E8</td>
<td>1.66E9</td>
<td>$O(n^r)$</td>
<td></td>
</tr>
<tr>
<td>MfemRT0</td>
<td>1.71E4</td>
<td>1.80E4</td>
<td>1.88E4</td>
<td>7.35E4</td>
<td>N/A</td>
<td></td>
</tr>
</tbody>
</table>

Figure 4: Example 2. (a) A center-low-permeability profile; (b) The numerical pressure and velocity obtained by DfvmP1 on a 40x40x2 triangular mesh.
**Example 3: Random permeability.** We consider a heterogenous example on the unit square \( \Omega = (0,1)^2 \) that was tested in [16]. The permeability \( K = K_E I_2 \), where \( I_2 \) is the order 2 identity matrix and \( K_E \) is a piecewise constant defined on a uniform \( 10 \times 10 \) rectangular mesh. The value of \( K_E \) on the rectangular blocks follows a log-normal random distribution. In other words, \( \log(K_E) \) has a mean 0 and a standard deviation 1, see Figure 5. In Figure 6, we can observe the 1st order and 2nd order convergence of the discrete max norms of the errors of the numerical pressures at the nodes on the left and right boundaries.

![Figure 5: Example 3. (a) A 10x10 random permeability profile; (b) The numerical pressure and velocity obtained by MfemRT0 on a 40x40x2 triangular mesh.](image)

![Figure 6: Example 3. Convergence rates of errors in nodal pressures for the three locally conservative methods. Left panel: errors of the nodal pressures on the left boundary of the domain; Right: errors of the nodal pressures on the right boundary of the domain.](image)
Table 4: Iteration numbers of the unpreconditioned and preconditioned DfvmP1WOP on Example 3 with \( n \times n \times 2 \) triangular meshes

<table>
<thead>
<tr>
<th>( n )</th>
<th>Condition number</th>
<th>Number of iterations for unpreconditioned</th>
<th>Number of iterations for preconditioned</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>1.85E4</td>
<td>144</td>
<td>41</td>
</tr>
<tr>
<td>20</td>
<td>2.57E5</td>
<td>759</td>
<td>171</td>
</tr>
<tr>
<td>40</td>
<td>3.91E6</td>
<td>2723</td>
<td>656</td>
</tr>
<tr>
<td>80</td>
<td>6.15E7</td>
<td>8202</td>
<td>2482</td>
</tr>
</tbody>
</table>

We list in Table 4 the iteration numbers of the unpreconditioned and preconditioned DfvmP1WOP. Here gmres is used with restart 20 and tolerance 1E-9 [17]. These results show that the diagonal preconditioner in Subsection 2.2 does help reduce iteration numbers significantly.

4. Concluding Remarks

It is clear from the comparison that both DfvmP1 and DfvmP1WOP produce numerical results comparable (higher order accuracy in pressure but the same order accuracy in velocity) to that of MfemRT0 by solving for roughly the same number of unknowns. However, the discrete linear systems generated by DfvmP1 and DfvmP1WOP are positive-definite (and symmetric in most cases) and hence easier to solve than the saddle-point linear systems obtained from MfemRT0. Among the two forms of the discontinuous finite volume method (without and with weak over-penalization), the latter is easier to implement and has no need for choosing any penalty factors. Although its condition number is large, but a simple block diagonal preconditioner is readily available [13]. We conclude that, for Darcy’s flows, the discontinuous finite volume method (both forms) could be a viable alternative to the classical mixed finite element method.

It should be pointed out that the errors in the above numerical experiments exhibit optimal convergence rates, but the existing theoretical results [1, 8, 13, 22] assume higher regularity than it should be for applications like Darcy’s flows. Error analysis for the discontinuous finite volume methods with minimal regularity requirements is challenging but currently under our investigation [12].

As is well known for porous media flows, the Darcy’s law is tightly connected with transport problems. A numerical velocity obtained from any of the above three locally conservative methods can be used in a transport solver. Due to the page limitation, we don’t address the coupling of the Darcy’s law and transport equations in this paper. The interested reader is referred to [16] for a detailed discussion on this issue.

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