Lenz-Barlotti I.4 Perspectivity Groups are Abelian

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Abstract: We extend a 1972 result of Kantor and Pankin and give a new elementary proof of the assertion in the title for projective planes of arbitrary order. The main tool appears in the very first book on group theory by Jordan in 1870.

The Lenz-Barlotti classification of projective planes is based on the possible configurations of point-line pairs for which Desargues theorem holds. Desargues theorem holds for such a pair (p, l) if and only if the plane admits a full group of perspectivities with center p and axis l. By definition, such a perspectivity group fixes all points on l, all lines on p and is maximally transitive consistent with these conditions. The Lenz-Barlotti *figure* of a projective plane Π is the set of point-line pairs for which Π is (p, l)-transitive and determines the Lenz-Barlotti *class* of Π. The plane Π is of class I.4 (respectively I.3) if its Lenz-Barlotti figure consists of the three non-incident point-line pairs of a triangle (respectively two of these pairs). These classes are two of the five for which existence questions remain open [1].[1] The purpose of this paper is to give a new elementary proof of:

Theorem 1 *Let Π be a projective plane of Lenz-Barlotti type I.4. Then its three transitive perspectivity groups are isomorphic and Abelian.*

Kantor and Pankin [5] prove this when Π is finite, using deep results of Suzuki regarding finite groups with subgroups that partition the group's non-identity elements.

For the group G, the *right* and *left regular representations* of G are group homomorphisms from G to $Sym(G)$ (the group of permutations of the elements of G), defined by the equations $h^{\lambda(g)} = g^{-1}h$ and $h^{\rho(g)} = hg,\ g, h \in G$. The main tool of our proof is quite elementary [2, p 86]:

Theorem 2 *(Jordan [4, p 60]) If $\sigma \in Sym(G)$ commutes with $\rho(g)$ for all $g \in G$, then $\sigma = \lambda(h)$ for some $h \in G$; this holds symmetrically with λ and ρ reversed.*

We use the permutation group notation of Weilandt [8]. In particular, suppose the group G acts on the set Ω, so Ω is a G-space. For $\omega \in \Omega$, $G_\omega := \{g \in G : \omega^g = \omega\}$.

Suppose G acts transitively on Ω. Then the map $\Omega \to G_\omega\backslash G := \{G_\omega g : g \in G\}$ given by $\omega^g \mapsto (G_\omega)g$ is a G-space isomorphism, and the set $G_\omega\backslash G$ of *right G_ω cosets in G* is an *internal realization* of Ω. Since $x^{-1}G_\omega x = G_{\omega^x}$, different choices for $\omega \in \Omega$ lead to isomorphic G-spaces.

An *incidence structure* is a triple (X, Y, I) where $I \subseteq X \times Y$. G is an *automorphism group* of (X, Y, I) if X and Y are G-spaces and the induced action of G on $X \times Y$ leaves I invariant.

[1] Existence of finite planes in either class is open. Naumann [7] constructs infinite planes in class I4.

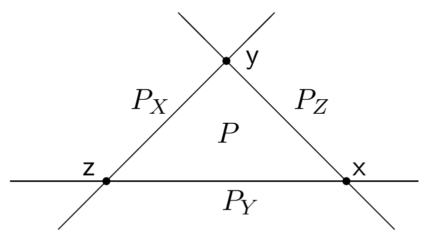
A *coset geometry* associated with the action of G on I is an isomorphic incidence structure consisting entirely of internal transitive G-spaces and G-invariant relations between them. Of course any coset geometry isomorphism involves synchronized G-space isomorphisms having domains the incidence structure parts X, Y and I

By building coset geometries for multiple incidence structures that admit the same abstract group, one can reveal subtle relationships between known combinatorial structures. When an incidence structure has a sufficiently rich automorphism group G, Theorem 2 allows the incidence relation to be expressed in terms of $\lambda : G \to Sym(G)$ as we do in section 2. Under the assumptions of Theorem 1 we show that another automorphism group Z is forced into $Im(\lambda)$ and there results a certain equality (equation (3)). The proof of Theorem 1 is completed by a combinatorial interpretation of this inequality using $\rho(G)$.

1 Geometric preliminaries

Let $\Pi = (P^*, L^*, F^*)$ be a projective plane with points P^*, lines L^* and incident point-line pairs (*flags*) F^*. Denote the unique line through the points x and y by xy and dually use the juxtaposition of labels for two lines to denote their point of intersection. In practice, there is no confusion about grouping.

Suppose that Π is of Lenz-Barlotti class at least I.3. This means there are three non-collinear points $\mathsf{x}, \mathsf{y}, \mathsf{z}$ so that there is a full $(\mathsf{x}, \mathsf{yz})$-perspectivity group X as well as a full $(\mathsf{y}, \mathsf{xz})$-perspectivity group Y. Set $G := \langle X, Y \rangle$ and observe that there are seven G-orbits on points which we label as indicated in the figure.



For example, P_X consists of the points of yz not in $\{\mathsf{y}, \mathsf{z}\}$. P_Y and P_Z are defined analogously, and P consists of all points off the triangle $\mathsf{x}\ \mathsf{y}\ \mathsf{z}$. We call elements of P *ordinary*. Points of the triangle which are not vertices are called *axial*, and they fall into the sets P_X, P_Y, and P_Z. *Ordinary* and *axial lines* are defined dually.

Lemma 1 *The groups X and Y commute and are normal in G. Moreover G is the internal direct product of X and Y and acts faithfully and regularly on both P and L.*

Proof. The first claim is a consequence of the fact that X is the kernel of the action of G on P_X and it acts regularly (sharply transitively) and faithfully on P_Y, while Y is the kernel of the action of G on P_Y and acts regularly while acting faithfully on P_X. Therefore, $x \in X$ and $y \in Y$ commute on $P_X \cup P_Y$.

The ordinary point p is uniquely the intersection of the axial line $p\mathsf{x}$ and $p\mathsf{y}$, so the images of the axial points $p\mathsf{xyz} \in P_x$ and $p\mathsf{yxz} \in P_y$ uniquely determine p^{xy}. The transitivity of G on P therefore follows geometrically from its transitivity on $P_X \times P_Y$.

The claim for L follows from the above argument applied to the dual plane. ∎

Lemma 2 *(Hughes [3]) The groups X and Y are isomorphic. If these groups are Abelian, then the kernel of the action of G on P_Z is $(\mathsf{z}, \mathsf{xy})$-transitive and Π is of type I.4.*

Proof. The groups X and Y both act regularly on P_Z and commute by Lemma 1. Therefore their action on P_Z is just as in Theorem 2, and they act as the left and right regular representations of the same group. This establishes the first claim. Suppose $X \cong Y$ is Abelian. Then Lemma 1 implies that $G = \langle X, Y \rangle$ is also Abelian. Take w to be any point on xy. Then each axial point on xy has the form w^x for some $x \in X$ and so the stabilizer in $G_{\mathsf{w}^x} = x^{-1} G_{\mathsf{w}} x = G_{\mathsf{w}}$, as G is Abelian. This implies that G_{w} is $(\mathsf{z}, \mathsf{xy})$-transitive. ∎

Two ordinary points do not determine an ordinary line if and only they determine an axial line. In fact there are three partitions π_X π_Y and π_Z of the ordinary points P with the property that two points are in the same part of one of these partitions if and only if they determine an axial line on x, y or z, respectively. There is also a dual partition of lines, and together with the *ordinary incidence structure* $(P, L, F := F^* \cap P \times L)$ they uniquely determine all incidences for the plane Π. Thus Π is uniquely determined by the incidence relation between the ordinary points and ordinary lines [1].

2 The ordinary incidence relation coset geometry

In this section we show that the incidence relation between ordinary points and ordinary lines appears in the internal coset geometry as the **left** regular representation of a certain subset Δ of G. When Π is finite, the set Δ is called a **neo-difference set** [1].

We are exclusively concerned with collineations of Π that fix each of $\mathsf{x}, \mathsf{y}, \mathsf{z}$. Call this group A, and note that it normalizes G. In order to build an A-invariant coset geometry from G for the ordinary incidence structure, a *seed ordinary point* and a *seed ordinary line* must be specified. To insure that the associated internal G-spaces are properly synchronized and that the coset geometry admits A as an automorphism group, we take the seed point to be incident with the seed line. Thus we specify a single seed ordinary flag (p, l). Take the p to be any ordinary point, but it is convenient to take the line l on p so that $\mathsf{w} := l\mathsf{xy}$ is in a nontrivial $H := G_{p\mathsf{z}}$-orbit if possible (see final figure).

Having specified a G-coset geometry for the incidence relation between ordinary points and lines, we are now in a position to express this incidence relation in terms of G alone. For $A, B \in \{P, L\}$ and $S \subseteq A \times B$, define

$$R_S := \{(g, h) \in G \times G : ((A \text{ seed})^g, (B \text{ seed})^h) \in S\} \subseteq G \times G.$$

Extend this notation to permutations $\pi \in Sym(G)$ by defining $R_\pi := \{(g, g^\pi) : g \in G\}$, so if $t \in G$ then $R_{\rho(t)} = \{(g, gt) : g \in G\}$ and $R_{\lambda(t)} = \{(g, t^{-1}g) : g \in G\}$.

For $a \in A$, the associated ordinary point permutation, $a|P$, and the associated ordinary line permutation, $a|L$, correspond to the following relations on G:

$$R_{a|P} = \{(g, h) : (p^g)^a = p^h\} \text{ and } R_{a|L} = \{(g, h) : (l^g)^a = l^h\}.$$

By the choice of l incident with p, these two relations on G coincide and equal $R_{\rho(a)}$.

Suppose $a, b \in G$ and $f = (p^a, l^b) \in P \times L$ is an ordinary flag. Since G acts regularly on P and on L, there is a unique flag in $f^{\,G} \subset F$ with point p. Consequently R_{f^G} is the table of values of the bijective function $ax \leftrightarrow bx$: $G \to G$. This function is $\lambda(ab^{-1})$ because:

$$R_{f^G} = \{(ax, bx) : x \in G\} = \{(y, (ab^{-1})^{-1}y) : y = ax \in G\} = R_{\lambda(ab^{-1})}.$$

But F is a disjoint union of G-orbits, each of which contains a unique flag having line l, so

$$R_F = \bigcup_{d \in \Delta} R_{\lambda(d)}, \text{ where } \Delta = \{d \in G : (p^d, l) \in F\}.$$

In other words, for $h \in G$, the set of ordinary points incident with l^h is $\{p^{dh} : d \in \Delta\}$, or simply that Δh is the set of labels for points incident with l^h.

Lemma 3 *Suppose $a \in A$. Then*

$$R_a \circ R_F = R_{a|P} \circ R_F = R_F \circ R_{a|L} = R_F \circ R_a. \tag{1}$$

Where $R_1 \circ R_2 := \{(g, k) \in G \times G : \exists h \in G$ so that $(g, h) \in R_1, (h, k) \in R_2\}$ is the composition of the relations R_1 and R_2.

Proof. The first and last equalities follow from the choice of l incident with p as already noted. By definition, the (g, h)-entry of the composition $R_{a|P} \circ R_F$ is 1 or 0 according to whether there is $k \in G$ such that $(p^g)^{a|P} = p^k$ and $(p^k, l^h) \in F$. But a is a collineation exactly when this situation occurs if only if there is $s \in G$ such that $(p^{\,g}, l^s) \in F$ and $(l^s)^{a|L} = l^h$. This in turn is equivalent to the condition that both the (g, s) entry of R_F and the (s, h) entry of $R_{a|P}$ are 1. The result follows. ■

3 Planes of type I.4

Finally, suppose Π is of type I.4 and that the $(\mathsf{z}, \mathsf{xy})$-transitive perspectivity group Z is present. By Lemma 1, Lemma 2 and the symmetry of the Lenz-Barlotti figure, Z acts on the ordinary points P as a permutation group that commutes with G, and G acts regularly on P. Also by Lemma 2, and by the symmetry of the Lenz-Barlotti figure, Z is isomorphic to both X and Y.

By yet another application of Theorem 2 and the careful choice of l, there is a monomorphism $\phi : Z \to G$ such that the point permutation z_P and line permutation z_L induced by $z \in Z$ are given by:

$$p^g \to p^{\phi(z)^{-1}g} \text{ and } l^g \to l^{\phi(z)^{-1}g}, \text{ or equivalently, } R_{z_P} = R_{\lambda(\phi(z))} = R_{z_L} \text{ for } z \in Z. \tag{2}$$

The action of G on the z-axial lines is realized in the internal coset-geometry by **right** multiplication on $G_{p\mathsf{z}} \backslash G$. By definition, Z acts on P leaving each of the z-axial lines invariant and so $\lambda \circ \phi(Z)$ acts trivially on $G_{p\mathsf{z}} \backslash G$. This is only possible because λ involves **left** multiplication in G, and it implies that $\phi(Z) = G_{p\mathsf{z}}$.

Take $z \in Z$ and use Lemma 3, with $a = z$, to rewrite equation (1) by applying equation (2) to obtain: $R_{\lambda(\phi(z))} \circ R_F = R_F \circ R_{\lambda(\phi(z))}$, for all $z \in Z$. This is equivalent to an equality of sets of permutations in $Sym(G)$:

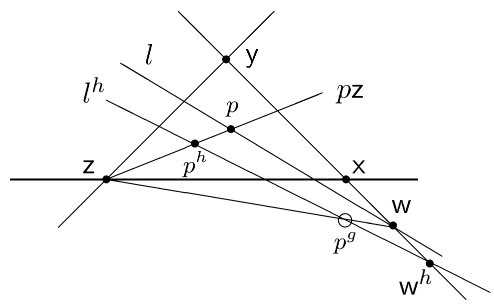
$$\bigcup_{d\in\Delta} \lambda(\phi(z))\lambda(d) = \lambda(\phi(z)) \left[\bigcup_{d\in\Delta} \lambda(d)\right] = \left[\bigcup_{d\in\Delta} \lambda(d)\right] \lambda(\phi(z)) = \bigcup_{d\in\Delta} \lambda(d)\lambda(\phi(z)).$$

But λ is a group isomorphism and we have just shown that $\phi(Z) = G_{\mathsf{pz}}$, so

$$h\Delta = \Delta h \text{ for all } h \in H := G_{\mathsf{pz}}. \tag{3}$$

The $(\mathsf{z}, \mathsf{xy})$-perspectivity group, Z, fixes w, so the Z-orbit of l contains no ordinary points on zw. Therefore no ordinary point on zw is labeled by an element of $\cup_{h\in H} h\Delta$.

By choice of w, $\mathsf{w}^h \neq \mathsf{w}$ for some $h \in H$. Therefore the ordinary points labeled by $\cup_{h\in H}\Delta h$ include all ordinary points on l^h. In particular, $l^h\mathsf{zw}$ is labeled by an element of G in $\cup_{h\in H}\Delta h$ (see figure). This contradicts equation (3) and completes the proof. ■



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