M261 Test 2 Review

- Know how to plot level curves and level surfaces for functions of two and three variables, respectively.
- Know how to find the domain and range of a function of two variables, and determine if the domain is open, closed, and/or bounded.
- Understand how to determine continuity of a function, and to find points of discontinuities.
- Use the two-path test to show that a limit of a function of two variables does not exist at a point.
- Compute partial derivatives
- Given an equation relating several variables, use implicit differentiation to compute partial derivatives.
- Given a relationship between several functions of several variables, use a tree diagram to derive a chain rule formula for a derivative.
- Compute directional derivatives. That is, given a function, initial point and a vector, find the rate of change of the function in the direction of the vector at some initial point.
- Find the equation for a tangent line to a level curve.
- Given a function, compute the directions of greatest change, greatest negative change, and no change, and find the magnitude of the greatest change.
- Given a level surface, find the equation of the tangent plane to a point.
- Given two level surfaces, find the tangent line of intersection of the surfaces at a point.
- Given a function of two or three variables, find the linearization of the function at that point.
- Use the linerization to approximate change in the function, given change in the independent variables.
- Use the directional derivative to approximate the change in a function in some direction.
- Compute critical points, and determine if they are local maxima, local minima, or saddle points.
- Compute the absolute maximum and minimum of a function on a closed bounded region.
- Use Lagrange multipliers to compute the maximum or minimum of a function f subject to some constraint $g = 0$.

A **level curve** of a function of two variables $f(x, y)$ is the 2D graph of the equation $f(x, y) = c$, where c is some fixed constant.

A **level surface** of a function of three variables $f(x, y, z)$ is the 3D graph of the equation $f(x, y, z) = c$, where c is some fixed constant.

- The set $\mathbb{R}$ denotes the set of real numbers.
- The set $\mathbb{R}^n$ denotes the set of n-tuples of real numbers, or the set of vectors of size n. (For example, $x = (1, 2, 3, 4)$ is a point in $\mathbb{R}^4$)
- An **open ball** in $\mathbb{R}^n$ centered at point x, of radius $r > 0$, is the set of all points y such that $|x - y| < r$ (In $\mathbb{R}^2$, open balls are circles, and in $\mathbb{R}^3$ they are spheres.)
- A point x is an **interior point** of a region R if it is contained in an open ball centered at x, where the ball is contained entirely inside of R.
- A point x is a **boundary point** of a region R if every open ball centered at x contains points in R and points not in R.
- A region R is **open** if every point in R is an interior point of R.
- A region R is **closed** if it contains all of its boundary points.
- A region is **bounded** if it is contained inside an open ball of finite radius, and is **unbounded** otherwise.

Two-Path Test For Nonexistence of a Limit If a function $f(x, y)$ has different limits along two different paths as (x, y) approaches (x_0, y_0), then $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ does not exist.

The **partial derivative** of $f(x, y)$ with respect to x is

$$\frac{\partial f}{\partial x} = f_x = \lim_{h\to 0} \frac{f(x + h, y) - f(x, y)}{h}.$$

The **partial derivative** of $f(x, y)$ with respect to y is

$$\frac{\partial f}{\partial x} = f_y = \lim_{h\to 0} \frac{f(x, y + h) - f(x, y)}{h}.$$

The **directional derivative** of f in the direction of the unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ is the function

$$D_{\mathbf{u}}f = \lim_{h\to 0} \frac{f(x + hu_1, y + hu_2) - f(x, y)}{h} = \nabla f \cdot \mathbf{u},$$

where the limit exists.

The direction of greatest rate of change occurs in the direction of ∇f and the greatest rate of change is $|\nabla f|$.

The direction of least rate of change (or greatest negative rate of change) occurs in the direction $-\nabla f$ and the value is again $|\nabla f|$.

The directions in which f does not change are orthogonal to the gradient. These can be found by solving $\nabla f \cdot \mathbf{u} = 0$ for $\mathbf{u}$.

Given a level surface $f(x, y, z) = c$, the tangent plane to the surface at (x_0, y_0, z_0) is given by

$$f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) = 0. \tag{1}$$

For a function of the form $f(x, y)$, the tangent plane to the surface at (x_0, y_0) is

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0). \tag{2}$$

To find the tangent line to the curve of intersection of two surfaces at a point $P_0(x_0, y_0, z_0)$, we take find the vector orthogonal to both tangent planes, $\mathbf{v} = \langle v_1, v_2, v_3 \rangle = \nabla f \times \nabla g$. Then, the tangent line is parallel to $\mathbf{v}$ and passes through P_0, so the equations are given by $x = x_0 + v_1 t, y = y_0 + v_2 t, z = z_0 + v_3 t$.

To estimate the change in f at P_0 in the direction of a unit vector $\mathbf{u}$, we use the approximation formula

$$\Delta f \approx (\nabla f|_{P_0} \cdot \mathbf{u})\Delta s,$$

where Δs is the distance moved from P_0 in the direction of $\mathbf{u}$.

The **linearization** of a function $f(x, y)$ at a point (x_0, y_0) is the function

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0). \tag{3}$$

Then $L \approx f$.

For a function $f(x, y)$, if $f_x, f_y, f_{xx}, f_{xy}, f_{yy}$ exist and are continuous on a rectangle R centered at (x_0, y_0), and if M is any upper bound for $|f_{xx}|, |f_{xy}|$, and $|f_{yy}|$ on R, then the linearization error $E(x, y) = L(x, y) - f(x, y)$ satisfies

$$|E(x, y)| \leq \frac{1}{2} M \left(|x - x_0| + |y - y_0|\right)^2. \tag{4}$$

The **linearization** of a function $f(x, y, z)$ at a point (x_0, y_0, z_0) is the function

$$L(x, y, z) = f(x_0, y_0, z_0) + f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0). \tag{5}$$

Then $L \approx f$.

For a function $f(x,y,z)$, if $f_x, f_y, f_z, f_{xx}, f_{yy}, f_{zz}, f_{xy}, f_{xz}, f_{yz}$ exist and are continuous on a 3D rectangle R centered at (x_0, y_0, z_0), and if M is any upper bound for $|f_{xx}|, |f_{yy}|, |f_{zz}|, |f_{xy}|, |f_{xz}|$, and $|f_{yz}|$ on R, then the linearization error $E(x,y,z) = L(x,y,z) - f(x,y,z)$ satisfies

$$|E(x,y,z)| \leq \frac{1}{2} M \left(|x - x_0| + |y - y_0| + |z - z_0|\right)^2 . \tag{6}$$

A critical point (x_0, y_0) of a function $f(x,y)$ either satisfies $f_x(x_0,y_0) = f_y(x_0,y_0) = 0$, or is a point for which f_x or f_y does not exist.

Second Derivative Test for Local Extreme Values Suppose that f has continuous first and second partial derivatives in a disk centered at (a,b) and that $f_x(a,b) = f_y(a,b) = 0$. Also suppose that $\det H(x,y) = f_{xx}f_{yy} - f_{xy}^2$. Then

1. f has a local maximum at (a,b) if $f_{xx} < 0$ and $\det H(f(a,b)) > 0$
2. f has a local minimum at (a,b) if $f_{xx} > 0$ and $\det H(f(a,b)) > 0$
3. f has a saddle point at (a,b) if $\det H(f(a,b)) < 0$
4. If $H(f(a,b)) = 0$ then this test is inconclusive.

To find a local maximum or local minimum of a function f with constraint $g = 0$, we solve the system

$$\nabla f = \lambda \nabla g \text{ and } g = 0.$$

To find a local maximum or local minimum of a function f with constraints $g_1 = 0$ and $g_2 = 0$, we solve the system

$$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2, g_1 = 0, g_2 = 0.$$

Practice Problems

1. Find the directional derivative of $f(x,y,z) = 3x^2y^2 + 2yz$ at (-1,0,4) in the direction of $\langle -1, 3, 3\rangle$.
2. Let $f(x,y) = x^2 - 4y^2, P_0(3,1)$.
   (a) Sketch the level curve of f passing through $(3,1)$.
   (b) Find the directional derivative of f at $P_0(3,1)$ in the direction of $\langle 3, -2\rangle$.
   (c) Find the direction of greatest increase of f at P_0, and plot this vector in your level curve sketch.

    (d) Find two directions in which f does not change at P_0.

3. Given that $z = xe^{-y} + ye^{-x}, x = u\sin v, y = v\cos u$, draw a tree diagram and find a chain rule for $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$, then find these derivatives.

4. Let $f(x, y) = x^3y + 2xy, P_0(1, 2)$.
    - (a) Find the tangent plane of f at P_0.
    - (b) Find the linearization L of f at P_0.
    - (c) Find an upper bound for the linearization error you found above on the rectangle $0.8 \leq x \leq 1.2, 1.8 \leq y \leq 2.2$.

5. Consider the surface $z^2 - 2xyz = x^2 + y^2$ and point $P_0(1, 2, -1)$.
    - (a) Find a vector normal to the surface at P_0.
    - (b) Find the tangent plane to the surface at P_0.

6. Find all local maxima, minima, and saddle points for the function $f(x, y) = x^3 + y^3 + 3x^2 - 3y^2 - 8$.

7. Find the absolute maxima and absolute minima of the function $f(x, y) = 2x^2 - 4x + y^2 - 4y + 1$ on the closed triangular region bounded by the lines $x = 0, y = 2, y = 2x$.

8. Find the global maximum and minimum of the function $f(x, y) = (x - 1)^2 + (y - 2)^2$ subject to the constraint $x^2 + y^2 = 45$.