M693 Geometric Data Analysis

Problem Set Six

Friday, May 10, 2002

You may work in groups of two (or one). If you work in a group of two please submit a single assignment. Both students will receive the same grade for the assignment. Please refrain from collaboration with students outside your group.

1 Theory

Solve the following problems:

1. Let $s = (s_1, \ldots, s_N) \in \{-1, 1\}^N$ be the spin–state of a neural network consisting of N neurons. Assume the coupling matrix is symmetric, $J_{ij} = J_{ji}$, $J_{ij} \in \mathbf{R}$, $1 \le i, j \le N$, and let

$$E(s) = -\frac{1}{8} \sum_{i,j=1}^{N} J_{ij} s_i s_j + \frac{1}{4} \sum_{i=1}^{N} \theta_i s_i$$

be the Hopfield energy function, where $\theta_i \in \mathbf{R}$, $1 \le i \le N$, are the components of the threshold vector. The associated membrane potential is defined by

$$h_i(s) = \frac{1}{2} \sum_{j=1}^{N} J_{ij} s_j - \theta_i.$$

For a given s let

$$s^{(i)} = (s_1, \ldots, s_{i-1}, -s_i, s_{i+1}, \ldots, s_N)$$

be the nearest neighbor of s obtained by flipping the ith spin. Show that

$$E(s^{(i)}) - E(s) = s_i h_i(s).$$

2. (a) Consider a neural network with $N = 3$ neurons, $s = (s_1, s_2, s_3)$. Show that it is not possible to find a symmetric coupling matrix J_{ij} and a threshold vector $(\theta_1, \theta_2, \theta_3)$ such that the four states $(-1, -1, -1)$, $(-1, 1, 1)$, $(1, -1, 1)$, $(1, 1, -1)$ are are all local minima of $E(s)$.

(b) Instead of spin variables consider Boolean variables $\sigma = (\sigma_1, \sigma_2, \sigma_3) \in$

$\{0,1\}^3$, where $\sigma_i = (s_i + 1)/2$. The state $\sigma_i = 0$ corresponds to 'off' of 'false', and the state $\sigma_i = 1$ to 'on' or 'true'. Identify the four states of (a) with a (well known) Boolean function $\sigma_3 = f(\sigma_1, \sigma_2)$ using the logical 'and' ($\wedge$), 'or' ($\vee$), and the inversion $\neg$.

Remark: The fact that the function of (b) could not be realized by a neural network caused neural network researchers in the eighties to introduce layers of hidden neurons.

3. Consider the discontinuous localization term $h(0) = 1$ and $h(x) = 0$ if $x \neq 0$ in Kohonen's SOM algorithm. What happens if a center is never a winner? Contrast this with the usual localization term $h(x) = \exp(-x^2/r^2)$.

4. Define a metric on a 2D index set that has the topology of a
   - a cylinder
   - a sphere
   - a torus

5. Consider the two-dimensional cluster centers $(1,2), (-2,3)$ and $(2,-1)$. Draw the (first order) Voronoi regions V_1, V_2 and V_3 as well as the second order Voronoi regions V_{12}, V_{13} and V_{23}.

2 Computing

Provide solutions including output and code to the following problems:

1. Modify the code to work for 1D data sets. Test your code on the data set $x_1 = .5, x_2 = .1, x_3 = .9, x_4 = .6, x_5 = .2$. Let the centers initially be the same as the data, i.e., $c_i = x_i$. Explain your answer (This is the question we posed in class. Be careful to optimize your solution.)

2. The problem considers an application of SOM's to visualization of data in $\mathbb{R}^{13}$ as points in the plane. The "Animals" data set (download off of the website) provides 13 attributes for each of 16 different animals. Using Kohonen's SOM algorithm map each of these animals to a square lattice in the plane. Interpret your results. You may use the sample code mysom.m on the class webpage as a starting point.