Problems

2.1 Show that the transformation

$$T\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}\right) = \begin{pmatrix} x_1 + x_2 \\ x_2 + x_3 \\ x_3 + x_4 \\ x_4 + x_1 \end{pmatrix}$$

is linear. Determine the matrix which represents this transformation.

2.2 Using the relationship $\mathbf{v}^{(i)} = \sum_{j=1}^{n} p_{ij}\mathbf{w}^{(j)}$ show that $\mathbf{y} = \mathbf{P}^T\mathbf{x}$ and deduce that $(\mathbf{P}^T)^{-1} = \mathbf{Q}^T$, i.e., the coordinate transformation is invertible.

2.3 Consider the vector $\mathbf{v}$ whose coordinates w.r.t. the basis $\mathcal{B}_1$ defined in example 2.6 are $(3, 5)$. What are the coordinates of $\mathbf{v}$ w.r.t. $\mathcal{B}_2$?

2.4 Let the basis $\mathcal{B}_1$ be the standard basis, i.e., $\mathbf{e}^{(1)} = (1\,0)^T, \mathbf{e}^{(1)} = (0\,1)^T$ and the basis $\mathcal{B}_2$ be given by the two vectors $\mathbf{v}^{(1)} = (1\,1)^T, \mathbf{v}^{(2)} = (-1\,1)^T$. Given $\mathbf{u}_{\mathcal{B}_1} = (1\,1)^T$ find $\mathbf{u}_{\mathcal{B}_2}$.

2.5 Let $\mathcal{B}_1$ be the standard basis and $\{\mathbf{w}^{(i)}\}$ be the vectors which define $\mathcal{B}_2$. Given $\mathbf{u}_{\mathcal{B}_2} = P^T\mathbf{u}_{\mathcal{B}_1}$ show $P^T = W^{-1}$ where $W = [\mathbf{w}^{(i)}|\ldots|\mathbf{w}^{(n)}]$.

2.6 Let the linear mapping L correspond to multiplication by the matrix A

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$$

which is given w.r.t. the basis $\mathcal{B}_1$ made up of the vectors $\mathbf{v}^{(1)} = (1\,1)^T$ and $\mathbf{v}^{(2)} = (1\,-1)^T$. Find the matrix A' which corresponds to the same mapping L but now w.r.t. the basis $\mathcal{B}_2$ made up of the vectors $\mathbf{w}^{(1)} = (1\,0)^T$ and $\mathbf{w}^{(1)} = (1\,2)^T$.

2.7 Define the union of two subspaces. Show that it is generally *not* a subspace.

2.8 Let W_1 and W_2 be vector spaces and $W = W_1 + W_2$. Show, by giving an example, that the decomposition of a vector $\mathbf{x} \in W$ is not unique, i.e.,

$$\mathbf{x} = \mathbf{w}_1 + \mathbf{w}_2 = \mathbf{w}_1' + \mathbf{w}_2'$$

where $\mathbf{w}_1 \neq \mathbf{w}_1'$, $\mathbf{w}_2 \neq \mathbf{w}_2'$ and $\mathbf{w}_1, \mathbf{w}_1' \in W_1$, $\mathbf{w}_2, \mathbf{w}_2' \in W_2$.

2.9 Let W be a subspace of the vector space V and let W_1 and W_2 be subspaces of W s.t. $W = W_1 + W_2$ and $W_1 \cap W_2 = \{0\}$. Show directly that any $\mathbf{x} \in W$ can be uniquely decomposed as

$$\mathbf{x} = \mathbf{w}_1 + \mathbf{w}_2$$

where $\mathbf{w}_1 \in W_1$ and $\mathbf{w}_2 \in W_2$.

2.10 Prove that if $W_1 \cup W_2$, then W_1 and W_2 are independent subspaces.

2.11 Consider the statements:

- Two independent subspaces must be orthogonal.
- Two orthogonal subspaces must be independent.

In each case, either prove or provide a counter example.

2.12 Prove that the column space of a matrix is a subspace.

2.13 Are perpendicular planes orthogonal?

2.14 Find the row space $\mathcal{R}(A^T)$ and left null space $\mathcal{N}(A^T)$ for the matrix $A = \mathbf{u}\mathbf{v}^T$.

2.15 Show that in general,

$$\mathbb{P}_{\mathbf{v}} = \frac{\mathbf{v}\mathbf{v}^T}{\mathbf{v}^T\mathbf{v}}$$

and

$$\mathbb{P}_{\mathbf{v}}^{\perp} = I - \frac{\mathbf{v}\mathbf{v}^T}{\mathbf{v}^T\mathbf{v}}$$

2.16 Let $V = \mathbb{R}^3$ and

$$\text{let } \mathbf{u}^{(1)} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \text{ and } \mathbf{u}^{(2)} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \text{ and let } \mathbf{x} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$$

and define $W_1 = \text{Span}(\mathbf{u}^{(1)}, \mathbf{u}^{(2)})$.

Find the orthogonal projection of $\mathbf{x}$ onto W_1. Also find the projection matrix $\mathbb{P}$ associated with this mapping.