2

Linear Spaces and Transformations

2.1 LINEAR TRANSFORMATIONS

Let V and W be two vector spaces. A mapping

$$L : V \to W$$

is said to be *linear* if

- $L(\mathbf{u} + \mathbf{v}) = L(\mathbf{u}) + L(\mathbf{v})$
- $L(c\mathbf{u}) = cL(\mathbf{u})$

for any vectors $\mathbf{u}, \mathbf{v} \in V$ and scalar $c \in \mathbb{R}$.

Example 2.1. Let A be an $m \times n$ matrix and define L_A

$$\begin{array}{rcl} L_A : \mathbb{R}^n & \to & \mathbb{R}^m \\ L_A(\mathbf{u}) & = & A\mathbf{u} \end{array}$$

Clearly L_A is a linear mapping as a consequence of the linearity of matrix multiplication.

Example 2.2. Differentiation, represented by the Jacobian matrix, is a linear mapping. See Section 3.9.5 in Chapter 3 for more details.

Example 2.3. Let $V = \mathbb{R}^3$ and $W = \mathbb{R}^2$. The mapping

$$\begin{array}{rcl} T : \mathbb{R}^3 & \to & \mathbb{R}^2 \\ T(x, y, z) & = & (x, y) \end{array}$$

is also linear.

While it is not surprising that matrix multiplication is a linear mapping, it is notable that *every* linear transformation between finite dimensional vector spaces may be represented as multiplication of a vector by an appropriate matrix. This representation is achieved by the introduction of a coordinate system, or basis for the space.

For example, the n vectors

$$\mathbf{e}^{(1)} = (1\,0\,\cdots\,0)^T$$

$$\mathbf{e}^{(2)} = (0\,1\,\cdots\,0)^T$$

$$\mathbf{e}^{(n)} = (0\,0\,\cdots\,1)^T$$

form a basis for $\mathbb{R}^n$ known as the *standard basis*. Thus any $\mathbf{u} \in \mathbb{R}^n$ can be written

$$\mathbf{u} = \alpha_1 \mathbf{e}^{(1)} + \alpha_2 \mathbf{e}^{(2)} + \cdots + \alpha_n \mathbf{e}^{(n)}.$$

The n-tuple $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ determines the *coordinates* of the point $\mathbf{u}$ w.r.t. to the standard basis.

We digress for a moment to emphasize the dependence of the coordinates of $\mathbf{u}$ on the choice of basis. For example, give another basis $\mathcal{B}$ for $\mathbb{R}^n$ consisting of the vectors $\{\mathbf{v}^{(1)}, \dots, \mathbf{v}^{(n)}\}$ we may represent $\mathbf{u}$ as

$$\mathbf{u} = x_1 \mathbf{v}^{(1)} + \cdots + x_n \mathbf{v}^{(n)}.$$

The n-tuple $(x_1, \dots, x_n)$ determine the coordinates of the point $\mathbf{u}$ w.r.t. to the new basis $\mathcal{B}$. More on this in the following section.

Now that the vector space is equipped with a basis we may make the connection between linear transformations and matrices.

Proposition 2.1. *Every linear mapping can be written as matrix multiplication.*

Proof. Consider the linear mapping

$$L : \mathbb{R}^n \to \mathbb{R}^m$$

Let $\mathbf{e}^{(1)}, \mathbf{e}^{(2)}, \dots, \mathbf{e}^{(n)}$ be the standard basis for $\mathbb{R}^n$. Furthermore, let $L(\mathbf{e}^{(j)}) = \mathbf{a}_j$ where $\mathbf{a}_j$ is a column vector in $\mathbb{R}^m$. Specifically, $L(\mathbf{e}^{(i)}) = (a_{1i}\, a_{2i}\, \dots\, a_{mi})^T$. Now let $\mathbf{u}$ be an arbitrary element of $\mathbb{R}^n$, i.e., $\mathbf{u} = \alpha_1 \mathbf{e}^{(1)} + \cdots + \alpha_n \mathbf{e}^{(n)}$. Thus we have

$$\begin{aligned} L(\mathbf{u}) &= L(\alpha_1 \mathbf{e}^{(1)} + \cdots + \alpha_n \mathbf{e}^{(n)}) \\ &= \alpha_1 L(\mathbf{e}^{(1)}) + \cdots + \alpha_n L(\mathbf{e}^{(n)}) \\ &= \alpha_1 \mathbf{a}_1 + \cdots + \alpha_n \mathbf{a}_n \\ &= A\alpha \end{aligned}$$

where $(A)_{ij} = a_{ij}$. □

Example 2.4. The matrix which corresponds to the linear operator of Example 2.3 is given by

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

2.2 CHANGE OF BASIS

A central issue in studying patterns is determining and utilizing the *correct* basis for a given set of data. Later we argue that empirical bases tend to be more efficient for representing specific data sets.

Motivated by this we now develop the basic mechanics of changing coordinate systems. To start, let $\{\mathbf{v}^{(i)}\}_{i=1}^n$ and $\{\mathbf{w}^{(i)}\}_{i=1}^n$ both be bases for $\mathbb{R}^n$, called $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively. Let $\mathbf{u}$ be an arbitrary element of $\mathbb{R}^n$. Thus in terms of the basis $\mathcal{B}_1$ we write

$$\mathbf{u} = x_1 \mathbf{v}^{(1)} + x_2 \mathbf{v}^{(2)} + \cdots + x_n \mathbf{v}^{(n)}$$

and in terms of $\mathcal{B}_2$ we write

$$\mathbf{u} = y_1\mathbf{w}^{(1)} + y_2\mathbf{w}^{(2)} + \cdots + y_n\mathbf{w}^{(n)}$$

giving the representation, or coordinates

$$\mathbf{v}_{\mathcal{B}_1} = (x_1 \ \ldots \ x_n)^T$$

w.r.t. $\mathcal{B}_1$ and coordinates

$$\mathbf{u}_{\mathcal{B}_2} = (y_1 \ \ldots \ y_n)^T$$

w.r.t. $\mathcal{B}_2$. Generally, the coordinate system which is in use is clear from the context and no specific reference is made to it.

By assumption, the $\{\mathbf{v}^{(i)}\}$ form a basis for $\mathbb{R}^n$, and any element in $\mathbb{R}^n$ can be expressed in terms of them. Thus, we may write

$$\mathbf{w}^{(i)} = \sum_{j=1}^{n} q_{ij}\mathbf{v}^{(j)}$$

which leads to

$$\begin{aligned}\mathbf{u} &= y_1(\sum_{j=1}^{n} q_{1j}\mathbf{v}^{(j)}) + \cdots + y_n(\sum_{j=1}^{n} q_{nj}\mathbf{v}^{(j)}) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} y_i q_{ij}\mathbf{v}^{(j)} = \sum_{j=1}^{n} x_j\mathbf{v}^{(j)}\end{aligned}$$

where $x_j = \sum_{i=1}^{n} q_{ij}y_i$ which is equivalent to

$$\mathbf{x} = Q^T\mathbf{y}$$

which we could equivalently write as $\mathbf{u}_{\mathcal{B}_1} = Q^T\mathbf{u}_{\mathcal{B}_2}$.

Alternatively, we can write $\mathbf{v}^{(i)} = \sum_{j=1}^{n} p_{ij}\mathbf{w}^{(j)}$ which leads to the relationship

$$\mathbf{y} = \mathbf{P}^T\mathbf{x} \tag{2.1}$$

from which it follows that $(\mathbf{P}^T)^{-1} = \mathbf{Q}^T$, i.e., the coordinate transformation is invertible.

Example 2.5. Given the basis vectors defining $\mathcal{B}_1$ to be

$$\mathbf{v}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{v}^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and that the basis vectors defining $\mathcal{B}_2$ are

$$\mathbf{w}^{(1)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{w}^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

find $\mathbf{u}_{\mathcal{B}_2}$ given

$$\mathbf{u}_{\mathcal{B}_1} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

We know that $\mathbf{u}_{\mathcal{B}_2} = P^T \mathbf{u}_{\mathcal{B}_1}$ so we must first compute P. By definition,

$$\mathbf{v}^{(i)} = p_{i1}\mathbf{w}^{(1)} + p_{i2}\mathbf{w}^{(2)}$$

so for $i = 1$ we have

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} p_{11} \\ p_{12} \end{pmatrix}$$

so

$$\begin{pmatrix} p_{11} \\ p_{12} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

From $i = 2$ it follows that

$$\begin{pmatrix} p_{21} \\ p_{22} \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

so

$$P^T = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$$

Given $\mathbf{u}_{\mathcal{B}_2} = P^T \mathbf{u}_{\mathcal{B}_1}$ we have

$$\mathbf{u}_{\mathcal{B}_2} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

The next question we address is how does the matrix A change with a change of basis. Assuming the matrix A is defined w.r.t. $\mathcal{B}_1$, what is the represenation A' for this matrix w.r.t. the new basis $\mathcal{B}_2$.

Let's first look at the action of the linear transformation A w.r.t. the basis $\mathcal{B}_1$, i.e.,

$$\mathbf{z} = A\mathbf{x}$$

Both $\mathbf{z}$ and $\mathbf{x}$ are coordinates w.r.t. $\mathcal{B}_1$. If we let $\mathbf{z}'$ and $\mathbf{x}'$ be the coordinates of $\mathbf{z}$ and $\mathbf{x}$ w.r.t. $\mathcal{B}_2$, respectively, then there exists a matrix M such that

$$\mathbf{z} = M\mathbf{z}' \quad \text{and} \quad \mathbf{x} = M\mathbf{x}'$$

So $z = Ax$ may be written $M\mathbf{z}' = AM\mathbf{x}'$ or

$$\mathbf{z}' = M^{-1}AM\mathbf{x}'$$

from which we conclude that

$$A' = M^{-1}AM$$

Thus $\mathbf{z}$ is the result of applying A to $\mathbf{x}$ in the first coordinate system and $\mathbf{z}'$ is the result of applying A' in the second coordinate system. In this case A and A' are said to be *similar* matrices.

Example 2.6. Given the bases as defined in Example 2.5 and that the mapping

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

is defined w.r.t. $\mathcal{B}_1$. What is the corresponding transformation w.r.t. $\mathcal{B}_1$? We saw previously that

$$M^{-1} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$$

so

$$A' = M^{-1}AM = \begin{pmatrix} -1 & 6 \\ 0 & 3 \end{pmatrix}$$

The question arises naturally, is there a coordinate system such that the action of a matrix is especially simple? The answer is yes for a large class of well-defined matrices. Suppose that the $n \times n$ matrix has n linearly independent eigenvectors. If these eigenvectors are chosen to be the columns of the transformation matrix M then the new matrix is diagonal, i.e.,

$$M^{-1}AM = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$$

where the λ_i are the eigenvalues associated with the independent eigenvectors. Note that if $A = A^T$, i.e., A is a symmetric matrix, then A is always diagonalizable. These ideas will be discussed further in Section 2.8.

2.3 OPERATIONS ON SUBSPACES

Given data sets lie intially within large vector spaces, it is important to be able to decompose such spaces into smaller ones. In this section we further develop our tools for decomposing patterns into especially useful subspaces. One of the main ideas to be developed is that of the projection matrix, but first, we examine the general problem of decomposing a vector space into the sum of independent subspaces.

Definition 2.1. *A subspace W of a vector space V is a subset of vectors such that*

- *if $\mathbf{w}, \mathbf{w}' \in W$ then $a\mathbf{w} + b\mathbf{w}' \in W$. In this case we say W is closed.*
- *$\mathbf{0} \in W$, i.e., every subspace must contain the zero vector.*

Proposition 2.2. *Define the set of vectors W*

$$W = \{\mathbf{w} : \mathbf{w} = \sum_i \alpha_i \mathbf{v}^{(i)}\}$$

The set W is a subspace. W is said to be spanned by the set of vectors $\{\mathbf{v}^{(i)}\}$.

Proof. Let $\mathbf{w}, \mathbf{w}' \in W$ so $\mathbf{w} = \sum_{i=1} c_i \mathbf{v}^{(i)}$ and $\mathbf{w}' = \sum_{i=1} c_i' \mathbf{v}^{(i)}$. It follows that

$$\mathbf{w} + \mathbf{w}' = \sum (c_i + c_i')\mathbf{v}^{(i)} \in W$$

$$a\mathbf{w} = \sum (ac_i)\mathbf{v}^{(i)} \in W$$

and lastly

$$\mathbf{0} = \sum_i 0\mathbf{v}^{(i)} \in W.$$

□

2.3.1 Intersection of Subspaces

Proposition 2.3. *If W_1 and W_2 are both subspaces, then so is their intersection $W_1 \cap W_2$.*

Proof. Let $\mathbf{x}, \mathbf{y} \in W_1 \cap W_2$, i.e., $\mathbf{x} \in W_1, W_2$ and $\mathbf{y} \in W_1, W_2 \Rightarrow a\mathbf{x} + b\mathbf{y} \in W_1$ and $a\mathbf{x} + b\mathbf{y} \in W_2$, in other words $a\mathbf{x} + b\mathbf{y} \in W_1 \cap W_2$. □

2.3.2 Addition of Subspaces

Let W_1 and W_2 be subspaces of a vector space V. The sum of these spaces is defined to be the result of taking all possible combinations of the elements of these two spaces.

Definition 2.2. *The sum of the vector subspaces W_1 and W_2 is written $W = W_1 + W_2$ and is defined to be the set*

$$W_1 + W_2 = \{\mathbf{w}_1 + \mathbf{w}_2 : \mathbf{w}_1 \in W_1, \mathbf{w}_2 \in W_2\}$$

The sum of three or more subspaces is defined analogously.

Proposition 2.4. *The sum of two subspaces is a subspace.*

Proof. Let $\mathbf{x}, \mathbf{y} \in W$, i.e., $\mathbf{x} = \mathbf{w}_1 + \mathbf{w}_2$ and $\mathbf{y} = \mathbf{w}_1' + \mathbf{w}_2'$ where $\mathbf{w}_i, \mathbf{w}_i' \in W_i$. Then we have $\alpha\mathbf{x} + \beta\mathbf{y} = \alpha(\mathbf{w}_1 + \mathbf{w}_1') + \beta(\mathbf{w}_2 + \mathbf{w}_2') = \alpha\mathbf{w}_1'' + \beta\mathbf{w}_2'' \in W_1 + W_2$. □.

The fact that the addition of two subspaces is a subspace provides us with a nice way to decompose a vector, i.e., if $\mathbf{x} \in W$ and $W = W_1 + W_2$ we can always write $\mathbf{x} = \mathbf{w}_1 + \mathbf{w}_2$ where $\mathbf{w}_i \in W_i$. After a little bit of experimenting with this decomposition it is clear that it is not unique. This ambiguity will generally be undesirable but can be avoided by restricting the relationship between W_1 and W_2 as described below.

2.3.3 Independence of Subspaces

To make the decomposition of a vector unique we require that the subspaces be independent.

Definition 2.3. *The subspaces W_1 and W_2 of V are independent if*

$$\mathbf{w}_1 + \mathbf{w}_2 = \mathbf{0}$$

implies

$$\mathbf{w}_1 = \mathbf{w}_2 = \mathbf{0}$$

where $\mathbf{w}_1 \in W_1$ and $\mathbf{w}_2 \in W_2$.

Independence ensures that the decomposition of V into subspaces is unique.

Proposition 2.5. *If W_1, W_2 are independent subspaces and $V = W_1 + W_2$, then the decomposition of $\mathbf{x} \in V$ given by*

$$\mathbf{x} = \mathbf{w}_1 + \mathbf{w}_2$$

is unique.

Proof. Let $\mathbf{x} = \mathbf{w}'_1 + \mathbf{w}'_2$ with each $\mathbf{w}'_i \in W_i$. Then

$$\mathbf{x} - \mathbf{x} = 0 = (\mathbf{w}_1 - \mathbf{w}'_1) + (\mathbf{w}_2 - \mathbf{w}'_2).$$

Since W_1 and W_2 are independent we conclude that $\mathbf{w}_i - \mathbf{w}'_i = 0$ or

$$\mathbf{w}_1 = \mathbf{w}'_1 \quad \text{and} \mathbf{w}_2 = \mathbf{w}'_2$$

□

Proposition 2.6. *If W_1, W_2 are independent subspaces then*

$$W_1 \cap W_2 = \{0\}.$$

Proof. Let $\mathbf{w} \in W_1 \cap W_2$. This implies that $\mathbf{w} \in W_1$ and $\mathbf{w} \in W_2$. Since W_2 is a subspace $-\mathbf{w} \in W_2$. Hence

$$\mathbf{w} + (-\mathbf{w}) = 0.$$

Since W_1 and W_2 are independent $\mathbf{w} = -\mathbf{w} = 0$.

Note that the converse is also true. See Problem 2.10.

2.3.4 Direct Sum Decompositions

From above we have that the independence of subspaces and the statement $W_1 \cap W_2 = \{0\}$ are equivalent. If either (equivalent) properties hold the decomposition is unique and we distinguish the decomposition from the mere addition of subspaces by writing

$$W = W_1 \oplus W_2$$

as the *direct sum decomposition* of W.

These ideas extend directly to the case of more than two subspaces. We cite the following important lemma from [19], p209.

Lemma 2.1. *Let V be a finite dimensional vector space. Let $W_1, \dots W_k$ be subspaces of V such that $W = W_1 + \cdots + W_k$. The following are equivalent:*

- *$W_1, \dots, W_k$ are independent.*
- *For each j, $2 \le j \le k$,*

$$W_j \cap \{W_1 + \cdots + W_{j-1}\} = \{0\}$$

- *If $\mathcal{B}_i$ is a basis for W_i then the collection of bases $\{\mathcal{B}_1, \dots, \mathcal{B}_k\}$ is a basis for W.*

Proof. See [19].

Furthermore, if any (and therefore all) of the above hold then the subspaces W_i form a direct sum decomposition of W which we write

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_k.$$

2.3.5 Orthogonal Direct Sum Decompositions.

A special but important instance of independent subspaces is orthogonal subspaces.

Definition 2.4. *A vector $\mathbf{v} \in V$ is said to be orthogonal to a subspace $W \subseteq V$ if $\mathbf{v}$ is orthogonal to every $\mathbf{w} \in W$. Two subspaces W_1 and W_2 are said to be orthogonal subspaces if for every $\mathbf{w}_1 \in W_1$ and $\mathbf{w}_2 \in W_2$ the inner product satsifies $(\mathbf{w}_1, \mathbf{w}_2) = 0$.*

Given a subspace W of the vector space V, the space of all vectors orthogonal to W in V is called the *orthogonal complement* of W written $W^\perp$.

Example 2.7. Let $V = \mathbb{R}^3$. Then the x-axis and y-axis are orthogonal subspaces of $\mathbb{R}^3$. Also, the orthogonal complement of the xy-plane is the $z-$ axis.

An important special case of the direct sum decomposition occurs when the subspaces are orthogonal. In this situation we distinguish the direct sum notation by writing $\dot{\oplus}$.

Example 2.8. Let V be an n-dimensional vector space with o.n. basis vectors $(\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \dots, \mathbf{v}^{(n)})$. If $W_i = \text{Span}(\mathbf{v}^{(i)})$ then we can write

$$V = W_1 \dot{\oplus} W_2 \dot{\oplus} \cdots \dot{\oplus} W_n$$

2.4 IMPORTANT SUBSPACES

In this section we describe the basic subspaces which be of use in what follows. It is implicit, unless otherwise state, that A is an $m \times n$ matrix.

Definition 2.5. *The range of A, denoted $\mathcal{R}(A)$, is the set of all vectors $\mathbf{v}$ such that $\mathbf{v} = A\mathbf{x}$ i.e.,*

$$\mathcal{R}(A) = \{\mathbf{v} \in \mathbb{R}^m \ : \ \mathbf{v} = A\mathbf{x} \textit{ for some } \mathbf{x} \in \mathbb{R}^n\}$$

The expression $\mathbf{v} = A\mathbf{x}$ may be rewritten

$$\begin{aligned}\mathbf{v} &= [\mathbf{a}^{(1)}|\mathbf{a}^{(2)}|\cdots|\mathbf{a}^{(n)}]\mathbf{x} \\ &= x_1\mathbf{a}^{(1)} + x_2\mathbf{a}^{(2)} + \cdots + x_n\mathbf{a}^{(n)}\end{aligned}$$

This expression reveals the fact that $\mathbf{v}$ lies in the span of the columns of A, i.e.,

$$\mathbf{v} \in \mathrm{Span}\{\mathbf{a}^{(1)}, \ldots, \mathbf{a}^{(n)}\}.$$

Hence the range of A, $\mathcal{R}(A)$, is also referred to as the *column space* of A.

Definition 2.6. *The null space of A, denoted $\mathcal{N}(A)$, is the set of all vectors $\mathbf{y}$ such that $A\mathbf{y} = 0$, i.e.,*

$$\mathcal{N}(A) = \{\mathbf{x} \in \mathbb{R}^n \ : \ A\mathbf{x} = 0\}$$

Definition 2.7. *The row space of A, denoted $\mathcal{R}(A^T)$, is the set of all vectors $\mathbf{x}$ such that $A\mathbf{x} = 0$, i.e.,*

$$\mathcal{R}(A^T) = \{\mathbf{x} \in \mathbb{R}^n \ : \ \mathbf{x} = A^T\mathbf{v} \textit{ for some } \mathbf{v} \in \mathbb{R}^m\}$$

Definition 2.8. *The left null space of A, $\mathcal{N}(A^T)$ is the set of all vectors $\mathbf{v}$ such that $A^T\mathbf{v} = 0$, i.e.,*

$$\mathcal{N}(A^T) = \{\mathbf{v} \in \mathbb{R}^m \ : \ A^T\mathbf{v} = 0\}$$

Example 2.9. Find the range $\mathcal{R}(A)$ and null space $\mathcal{N}(A)$ of the matrix

$$A = \mathbf{u}\mathbf{v}^T$$

To determine the range, rewrite the matrix

$$A = [v_1\mathbf{u}|\cdots|v_n\mathbf{u}]$$

from which it is apparent that

$$\mathcal{R}(A) = \alpha\mathbf{u}$$

Let $\mathbf{x}$ be an element of the null space, i.e.,

$$\mathbf{u}(\mathbf{v}^T\mathbf{x}) = 0$$

From the manner in which this is written we see that, since $\mathbf{u} \neq 0$,

$$\mathcal{N}(A) = \{\mathbf{x} : \mathbf{v}^T\mathbf{x} = 0\}$$

Proposition 2.7. *For any $m \times n$ matrix*

$$\mathcal{N}(A) \perp \mathcal{R}(A^T)$$

i.e., they are orthogonal subspaces of $\mathbb{R}^n$ and

$$\mathcal{N}(A^T) \perp \mathcal{R}(A)$$

i.e., they are orthogonal subspaces of $\mathbb{R}^m$.

The range, or column space of an $m \times n$ matrix A determines a subspace of $\mathbb{R}^m$. The number of independent vectors in this subspace, i.e., its dimension, is a very special and useful quantity for a matrix known as its *rank.*

Definition 2.9. *The column rank (row rank) of a matrix is defined as the number of independent columns (rows) in the column space $\mathcal{R}(A)$ (row space $\mathcal{R}(A^T)$).*

Proposition 2.8. *The row rank is equal to the column rank. In summary,*

$$r = \dim \mathcal{R}(A) = \dim \mathcal{R}(A^T)$$

Example 2.10. The matrix $A = \mathbf{u}\mathbf{v}^T$ has rank $r = 1$.

Proposition 2.9. *If A is an $m \times n$ matrix then*

$$r \leq \min(m, n)$$

Proposition 2.10.

$$r + \dim \mathcal{N}(A) = n$$

2.5 PROJECTION MATRICES

The direct sum provides a framework within which a vector space may be systematically split into subspaces that provide a unique expression for the decomposition of any vector in the space. In this Section we describe a procedure for constructing a mapping which takes a vector and executes this decomposition. Specifically, we refer to a matrix $\mathbb{P}$ as a *projection matrix* if

$$\mathbb{P}^2 = \mathbb{P}.$$

Such matrices are also said to be *idempotent.* See Figure 2.1 which depicts the possible actions of a projection matrix.

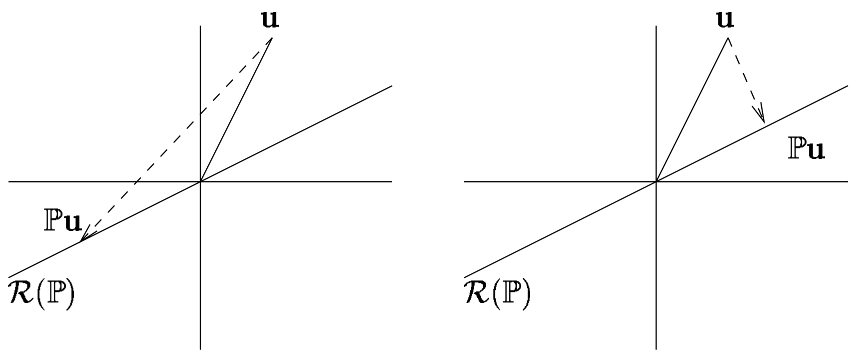

Fig. 2.1 Left: A nonorthogonal, or *oblique* projection. Right: an orthogonal projection.

Example 2.11. It is easy to verify that the matrix

$$\mathbb{P} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ 1 & \frac{1}{2} \end{pmatrix}$$

is a projection matrix. Note that it has rank 1 and that

$$\mathcal{R}(\mathbb{P}) = \alpha \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

2.5.1 Invariant Subspaces

Definition 2.10. *Let V be a vector space and L a linear operator on V. If W is a subspace of V, we say W is invariant under L if for each $\mathbf{w} \in W$ we have $L\mathbf{w} \in W$. In other words $L(W) \subseteq W$.*

If W_1 and W_2 are subspaces invariant under A (where A is the matrix that corresponds to the linear operator L) with $V = W_1 \oplus W_2$ then we say A is *reduced* or *decomposed* by W_1 and W_2.

We now show that a projection matrix naturally produces an invariant subspace.

Proposition 2.11.

$$\mathbf{v} \in \mathcal{R}(\mathbb{P}) \quad \textit{if and only if} \quad \mathbb{P}\mathbf{v} = \mathbf{v}$$

Proof. First assume $\mathbf{v} \in \mathcal{R}(\mathbb{P})$, i.e., $\mathbf{v} = \mathbb{P}\mathbf{x}$ for some $\mathbf{x} \in \mathbb{R}^n$. So $\mathbb{P}\mathbf{v} = \mathbb{P}^2\mathbf{x}$ but $\mathbb{P}^2\mathbf{x} = \mathbb{P}\mathbf{x} = \mathbf{v}$ from which we conclude that $\mathbb{P}\mathbf{v} = \mathbf{v}$. To prove the converse, assume that $\mathbb{P}\mathbf{v} = \mathbf{v}$. It follows directly that $\mathbf{v} \in \mathcal{R}(\mathbb{P})$. □

2.5.1.1 The Nullspace of $\mathbb{P}$. What is $\mathcal{N}(\mathbb{P})$? A member of this set is readily seen to be the difference between the the vector $\mathbf{v}$ being projected and its projection $\mathbb{P}\mathbf{v}$. If $\mathbf{v} = \mathbf{r} + \mathbb{P}\mathbf{v}$ then we have

$$\mathbf{r} = \mathbf{v} - \mathbb{P}\mathbf{v} \tag{2.2}$$

Projecting this vector $\mathbf{r}$ gives

$$\mathbb{P}\mathbf{r} = \mathbb{P}\mathbf{v} - \mathbb{P}^2\mathbf{v} = 0$$

So we conclude that $\mathbf{r} \in \mathcal{N}(\mathbb{P})$ and that the null space is unvariant under $\mathbb{P}$.

2.5.1.2 The Complementary Projection $I - \mathbb{P}$. Factoring the projection matrix in equation (2.2) produces

$$\mathbf{r} = (I - \mathbb{P})\mathbf{v} \tag{2.3}$$

so we see the natural decomposition

$$\mathbf{v} = \mathbb{P}\mathbf{v} + (I - \mathbb{P})\mathbf{v} \tag{2.4}$$

Note that given $\mathbb{P}$ is a projection matrix it follows that so is $I - \mathbb{P}$ since

$$\begin{aligned} (I - \mathbb{P})^2 &= I - 2\mathbb{P} + \mathbb{P}^2 \\ &= I - 2\mathbb{P} + \mathbb{P} \\ &= I - \mathbb{P} \end{aligned}$$

It follows that equation (2.4) represents a decomposition into invariant subspaces. Additionaly, we may employ the notation $\mathbb{Q} = I - \mathbb{P}$ to represent the projection matrix onto the null space.

Proposition 2.12.

$$\mathcal{R}(\mathbb{Q}) = \mathcal{N}(\mathbb{P}) \tag{2.5}$$

and

$$\mathcal{R}(\mathbb{P}) = \mathcal{N}(\mathbb{Q}) \tag{2.6}$$

Proof. We prove the first statement. First let $\mathbf{v} \in \mathcal{R}(\mathbb{Q})$ and show $\mathbf{v} \in \mathcal{N}(\mathbb{P})$. Given $\mathbf{v} = (I - \mathbb{P})\mathbf{x}$ for some $\mathbf{x}$ it follows $\mathbb{P}\mathbf{v} = (\mathbb{P} - \mathbb{P})x = 0$ so $\mathbf{v} \in \mathcal{N}(\mathbb{P})$ for an arbitrary $\mathbf{v}$ so we conclude $\mathcal{R}(\mathbb{Q}) \subset \mathcal{N}(\mathbb{P})$. Now let $\mathbf{v} \in \mathcal{N}(\mathbb{P})$ and show $\mathbf{v} \in \mathcal{R}(\mathbb{Q})$. If $\mathbb{P}\mathbf{v} = 0$, then $(I - \mathbb{P})\mathbf{v} = \mathbf{v}$ so $\mathbf{v} \in \mathcal{R}(\mathbb{Q})$. Again, since $\mathbf{v}$ is arbitrary, it follows that $\mathcal{N}(\mathbb{P}) \subset \mathcal{R}(\mathbb{Q})$. These results, taken together, prove the result that $\mathcal{R}(\mathbb{Q}) = \mathcal{N}(\mathbb{P})$. The second statement can be demonstrated in a similar fashion. □

2.5.1.3 Independence

Proposition 2.13.

$$\mathcal{R}(\mathbb{P}) \cap \mathcal{N}(\mathbb{P}) = \{0\} \tag{2.7}$$

Proof. From equation (2.5) we know $\mathcal{N}(\mathbb{P}) = \mathcal{R}(I - \mathbb{P})$. Let $\mathbf{v} \in \mathcal{R}(I - \mathbb{P})$, i.e., $\mathbf{v} = (I - \mathbb{P})\mathbf{x}$ for some $\mathbf{x}$. So $\mathbb{P}\mathbf{v} = 0$. But, by proposition 2.11, $\mathbf{v} \in \mathcal{R}(\mathbb{P})$ if and only if $\mathbb{P}\mathbf{v} = \mathbf{v}$, hence we conclude $\mathbf{v} = 0$ is the only element common to both $\mathcal{R}(\mathbb{P})$ and $\mathcal{N}(\mathbb{P})$. □

From these results it is now clear that a projection matrix separates a space into the sum of two independent subspaces. We recall that this is exactly the direct sum decomposition so we may write

$$V = \mathcal{R}(\mathbb{P}) \oplus \mathcal{N}(\mathbb{P})$$

It is also interesting to note that for every splitting

$$V = W_1 \oplus W_2$$

there exists a projection operator $\mathbb{P}$ such that

$$\mathcal{R}(\mathbb{P}) = W_1$$

and

$$\mathcal{N}(\mathbb{P}) = W_2$$

For details see [42].

2.6 ORTHOGONAL PROJECTION MATRICES

We have seen that projection matrices permit the decomposition of a space into subspaces. The most useful application of this idea is when the resulting subspaces are orthogonal, i.e., when the projection matrix and is complement produce orthogonal vectors. We begin with a basic definition.

Definition 2.11. *Let* $\mathbf{x} = \mathbf{w}_1 + \mathbf{w}_2$ *and* $\mathbf{w}_1 \in W_1, \mathbf{w}_2 \in W_2$ *with* $W_1 \perp W_2$. *The vector* $\mathbf{w}_1$ *is called the orthogonal projection of* $\mathbf{x}$ *onto* W_1 *and* $\mathbf{w}_2$ *is called the orthogonal projection of* $\mathbf{x}$ *onto* W_2.

Associated with an orthogonal projection is the operator, which we now refer to as an *orthogonal projection matrix*, which performs the projection described in the definition above. (Note that the orthogonal projection matrix should not be confused with an orthogonal matrix.)

Definition 2.12. *If the subspaces* $\mathcal{R}(\mathbb{P})$ *and* $\mathcal{N}(\mathbb{P})$ *are orthogonal, then the projection matrix* $\mathbb{P}$ *is said to be an orthogonal projection matrix.*

If $\mathbb{P}$ is an orthogonal projection matrix, then we may write the direct sum decomposition of the space as

$$V = \mathcal{R}(\mathbb{P}) \dot{\oplus} \mathcal{N}(\mathbb{P}).$$

Why are orthogonal projection matrices to be preferred over plain projection matrices?

2.6.1 Best Approximation Theorem

Suppose W is a subspace of an inner product space V and let $\mathbf{x} \in V$ be an arbitrary vector. One of goals is to find the best approximation to $\mathbf{x}$ by vectors in W. In other words, we seek a vector $\mathbf{w} \in W$ such that $\|\mathbf{x} - \mathbf{w}\|$ is a minimum.

Definition 2.13. *A best approximation to* $\mathbf{x}$ *by vectors in* W *is a vector* $\mathbf{w} \in W$ *such that*

$$\|\mathbf{x} - \mathbf{w}\| \leq \|\mathbf{x} - \mathbf{w}'\|$$

for all $\mathbf{w}' \in W$.

Theorem 2.1. The Projection Theorem: *Of all decompositions of the form*

$$\mathbf{x} = \mathbf{w}_1' + \mathbf{w}_2'$$

with $\mathbf{w}_1' \in W_1$ *the orthogonal projection provides the best approximation to* $\mathbf{x}$. *Equivalently, the orthogonal projection minimizes* $\|\mathbf{w}_2'\|$.

Proof. We rewrite

$$\begin{aligned}
\|\mathbf{x} - \mathbf{w}_1'\|^2 &= \|\mathbf{x} - \mathbf{w}_1 + \mathbf{w}_1 - \mathbf{w}_1'\|^2 \\
&= (\mathbf{x} - \mathbf{w}_1 + \mathbf{w}_1 - \mathbf{w}_1', \mathbf{x} - \mathbf{w}_1 + \mathbf{w}_1 - \mathbf{w}_1') \\
&= (\mathbf{x} - \mathbf{w}_1, \mathbf{x} - \mathbf{w}_1 + \mathbf{w}_1 - \mathbf{w}_1') + (\mathbf{w}_1 - \mathbf{w}_1', \mathbf{x} - \mathbf{w}_1 + \mathbf{w}_1 - \mathbf{w}_1') \\
&= (\mathbf{x} - \mathbf{w}_1, \mathbf{x} - \mathbf{w}_1) + (\mathbf{w}_1 - \mathbf{w}_1', \mathbf{w}_1 - \mathbf{w}_1') + 2(\mathbf{x} - \mathbf{w}_1, \mathbf{w}_1 - \mathbf{w}_1') \\
&= \|\mathbf{x} - \mathbf{w}_1\|^2 + \|\mathbf{w}_1 - \mathbf{w}_1'\|^2 + 2(\mathbf{x} - \mathbf{w}_1, \mathbf{w}_1 - \mathbf{w}_1')
\end{aligned}$$

Observe that

$$\mathbf{x} - \mathbf{w}_1 = \mathbf{w}_2 \in W_2$$

and that $\mathbf{w}_1 - \mathbf{w}_1' \in W_1$

$$\Rightarrow (\mathbf{x} - \mathbf{w}_1, \mathbf{w}_1 - \mathbf{w}_1') = 0$$

given that the projection is orthogonal. From this we have

$$\|\mathbf{x} - \mathbf{w}_1'\|^2 \geq \|\mathbf{x} - \mathbf{w}_1\|^2$$

in other words, $\mathbf{w}_1$ is the *best approximation* to $\mathbf{x}$. Note that $\|\mathbf{w}_2'\| = \|\mathbf{x} - \mathbf{w}_1'\|$ is a minimum for $\mathbf{w}_1'$ and since $\mathbf{w}_2 = \mathbf{x} - \mathbf{w}_1$ it follows that $\mathbf{w}_2' = \mathbf{w}_2$ in the case of the best approximation. □ Furthermore, it can be shown that this best approximation is unique, see [19] for details. In addition, these results may be extended to the general setting of metric spaces [36].

2.6.2 Criterion for Orthogonal Projections

Proposition 2.14. *If*

$$\mathbb{P} = \mathbb{P}^T \tag{2.8}$$

then the projection matrix is orthogonal.

Proof. Let $\mathbb{P} = \mathbb{P}^T$. $\mathbb{P}\mathbf{x} \in \mathcal{R}(\mathbb{P})$ and $(I - \mathbb{P})\mathbf{x} \in \mathcal{N}(\mathbb{P})$.

$$\begin{aligned}(\mathbb{P}\mathbf{x})^T(I - \mathbb{P})\mathbf{x} &= \mathbf{x}^T\mathbb{P}^T(I - \mathbb{P})\mathbf{x} \\ &= \mathbf{x}^T(\mathbb{P} - \mathbb{P}^2)\mathbf{x} \\ &= \mathbf{0}\end{aligned}$$

□

The converse of the above proposition is also true, i.e., if $\mathbb{P}$ is an orthogonal projection matrix, then $\mathbb{P} = \mathbb{P}^T$.

Example 2.12. It is easy to verify that the matrix

$$\mathbb{P} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

is an orthogonal projection matrix. Note that it has rank 1 and that

$$\mathcal{R}(\mathbb{P}) = \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Example 2.13. Every matrix of the form $\mathbf{v}\mathbf{v}^T$ is an orthogonal projection matrix if $\|\mathbf{v}\| = 1$.

$$\begin{aligned}(\mathbf{v}\mathbf{v}^T)^2 &= (\mathbf{v}\mathbf{v}^T)(\mathbf{v}\mathbf{v}^T) \\ &= \mathbf{v}(\mathbf{v}^T\mathbf{v})\mathbf{v}^T \\ &= \mathbf{v}\mathbf{v}^T\end{aligned}$$

Note that this projection matrix is rank one and that $\mathcal{R}(\mathbf{v}\mathbf{v}^T) = \text{Span}(\mathbf{v})$.

From this example we observe that any vector $\mathbf{u}$ may be orthogonally projected onto a given vector $\mathbf{v}$ by defining

$$\mathbb{P}_{\mathbf{v}}\mathbf{u} = (\mathbf{v}\mathbf{v}^T)\mathbf{u} = \mathbf{v}(\mathbf{v}^T\mathbf{u})$$

Also, the orthogonal complement, or residual $\mathbf{r}$ is then found to be

$$\begin{aligned}\mathbf{r} = \mathbb{P}_{\mathbf{v}}^{\perp} &= (I - \mathbb{P}_{\mathbf{v}})\mathbf{u} \\ &= \mathbf{u} - (\mathbf{v}^T\mathbf{u})\mathbf{v}\end{aligned}$$

We can leverage our ability to project $\mathbf{u}$ onto a single vector $\mathbf{v}$ into a method for computing the orthogonal projection of $\mathbf{u} \in \mathbb{R}^n$ onto a subspace W. To begin, we assume that we have an o.n. basis for the space $\mathbf{W}$ consisting of the vectors $\{\mathbf{w}^{(1)}, \ldots, \mathbf{w}^{(k)}\}$. We may view each of the $\mathbf{w}^{(i)}$ as spanning a one-dimensional subspace W_i. Clearly, each of these spaces is orthogonal, i.e.,

$$W_i \perp W_j, \quad i \neq j$$

Furthermore, the sum of these subspaces spans a k-dimensional subspace

$$W = W_1 + \cdots + W_k$$

From our previous deliberations,

$$W = W_1 \dot{\oplus} \ldots \dot{\oplus} W_k$$

In other words, the o.n. basis induces a direct sum decomposition of the subspace W. A projection onto W may be constructed from projections onto the individual supspaces.

The projection of $\mathbf{u}$ onto the i'th subspace space is given by

$$\mathbb{P}_{\mathbf{w}^{(i)}}\mathbf{u} = \mathbf{w}^{(i)}{\mathbf{w}^{(i)}}^T\mathbf{u}$$

If we write $\mathbb{P}_i \equiv \mathbb{P}_{\mathbf{w}^{(i)}}$, then the projection matrix onto W is given by

$$\mathbb{P} = \sum_{i=1}^{k} \mathbb{P}_i = \sum_{i=1}^{k} \mathbf{w}^{(i)}{\mathbf{w}^{(i)}}^T$$

Given the matrix $M = [\mathbf{w}^{(1)}|\ldots|\mathbf{w}^{(k)}]$, it follows

$$\mathbb{P} = MM^T. \tag{2.9}$$

2.6.3 Orthogonalization

In the course of the above computations we assumed that the subspace on which we were to project was equipped with an orthonormal basis. We now review the *Gram-Schmidt* procedure for computing an o.n. basis starting from a linearly independent set of vectors $\{\mathbf{v}^{(i)}\}_{i=1}^m$. Take as the first element

$$\mathbf{u}^{(1)} = \frac{\mathbf{v}^{(1)}}{\|\mathbf{v}^{(1)}\|} \tag{2.10}$$

The second element of this set is constructed using the same 2-to-1-dimensional projection technique discussed previously. The projection of $\mathbf{v}^{(2)}$ onto $\mathbf{u}^{(1)}$ is given by

$$\mathbb{P}_{\mathbf{u}^{(1)}}\mathbf{v}^{(2)} = (\mathbf{u}^{(1)}{\mathbf{u}^{(1)}}^T)\mathbf{v}^{(2)}$$

so the vector pointing orthogonally to $\mathbf{u}^{(1)}$ is the residual

$$\mathbf{r} = (I - \mathbb{P}_{\mathbf{u}^{(1)}})\mathbf{v}^{(2)}$$

Simplifying and normalizing this vector gives

$$\mathbf{u}^{(2)} = \mathbf{v}^{(2)} - (\mathbf{u}^{(1)^T}\mathbf{v}^{(2)})\mathbf{u}^{(1)} \tag{2.11}$$

Proceeding in the same fashion with the j'th direction we have

$$\mathbf{u}^{(j)} = \frac{\mathbf{v}^{(j)} - \sum_{i=1}^{j-1}(\mathbf{v}^{(j)}, \mathbf{u}^{(i)})\mathbf{u}^{(i)}}{\|\mathbf{v}^{(j)} - \sum_{i=1}^{j-1}(\mathbf{v}^{(j)}, \mathbf{u}^{(i)})\mathbf{u}^{(i)}\|}$$

Note that if the added direction $\mathbf{v}^{(j)}$ is dependent on the previous vectors then $\mathbf{u}^{(j)} = 0$.

Example 2.14. Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

Find the orthogonal projection matrix which takes an element of $\mathbb{R}^4$ onto $\mathcal{R}(A)$. Define $\mathbf{a}^{(1)} = (1\,0\,1\,0)^T$ and $\mathbf{a}^{(2)} = (1\,0\,0\,1)^T$. Given the 3rd column is a multiple of the first $\mathcal{R}(A) = \text{Span}(\mathbf{a}^{(1)}, \mathbf{a}^{(2)})$. To find the projection matrix $\mathbb{P}$ which maps an element of $\mathbb{R}^4$ onto $\mathcal{R}(A)$ we first determine an orthonormal basis for $\mathcal{R}(A)$. Clearly the columns $\mathbf{a}^{(1)}$ and $\mathbf{a}^{(2)}$ are linearly independent but they are not orthogonal. Using the Gram-Schmidt procedure we obtain

$$\mathbf{u}^{(1)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \end{pmatrix}^T$$

and

$$\mathbf{u}^{(2)} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & -1 & 2 \end{pmatrix}^T$$

The projection matrix onto $\mathbf{u}^{(1)}$ is given by

$$\mathbb{P}_1 = \mathbf{u}^{(1)}\mathbf{u}^{(1)^T} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and the projection matrix onto $\mathbf{u}^{(2)}$ is given by

$$\mathbb{P}_2 = \mathbf{u}^{(2)}\mathbf{u}^{(2)^T} = \frac{1}{6}\begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & -2 \\ 2 & 0 & -2 & 4 \end{pmatrix}$$

From this we have the projection matrix

$$\mathbb{P} = \mathbb{P}_1 + \mathbb{P}_2 = \frac{1}{3}\begin{pmatrix} 2 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & -1 \\ 1 & 0 & -1 & 2 \end{pmatrix}$$