4

Additional Theory and Applications of the KL Expansion

In this Chapter we continue our study of the K-L procedure and apply it in detail to a variety of problems, including the study of patterns which evolve in time, i.e., spatio-temporal data.

n Section 4.1 we develop the continuous version of the KL transform. This is important for theoretical reasons and will be used later Sections. In particular, in Section 4.3 we demonstrate how both the continuous and discrete expansions may be extended to have even and odd symmetry. In addition, in Section 4.2 the procedure is extended to both continuous and discrete vector functions.

The last major Section of this comprehensive Chapter on optimal expansions develops the local KL procedure. Of particular importance in Section 4.7 is the application of the method to computing local dimensionalities using the scaling properties of the singular values.

4.1 THE CONTINUOUS KL TRANSFORM

In many cases of importance it is easier to demonstrate theoretical results concerning the Karhunen-Loève procedure if the continuous form of the expansion is used. The derivation now moves to a function space setting, but remains essentially analogous to the derivation for the discrete case.

Now we assume that the data points $\{u^{(\mu)}(x)\}_{\mu=1}^{P}$ of our ensemble are functions which reside in the Hilbert space $L^2(a,b)$, $\infty \leq a < b \leq \infty$, i.e., the space of Lebesgue measurable functions

$$f : (a,b) \rightarrow \mathbb{C}$$

which are square-integrable

$$\int_a^b |f(x)|^2 dx < \infty$$

with the inner product

$$(f,g) = \int_a^b f(x)\overline{g(x)}dx \tag{4.1}$$

and induced norm

$$\|f\|^2 = (f,f).$$

The presentation here will not be technically exacting in the sense that sets of measure zero will be ignored. For a more mathematically detailed approach see [72]. In addition, we will assume that all of our functions are real.

In this infinite dimensional inner product space, we again seek to construct an optimal basis $\mathcal{B}$ with elements $\{\phi^{(j)}(x)\}_{j=1}^{\infty}$ in L^2. Our basis should be o.n., i.e.,

$$(\phi^{(i)}, \phi^{(j)}) = \int_a^a \phi^{(i)}(x)\phi^{(j)}(x)dx = \delta_{ij}.$$

Then, any square-integrable function can be expanded as

$$u^{(\mu)}(x) = \sum_{i=1}^{\infty} a_i^{(\mu)} \phi^{(i)}(x)$$

where the expansion coefficients $a_i^{(\mu)}$ are given by

$$a_i^{(\mu)} = (u^{(\mu)}, \phi^{(i)}) = \int_{-\infty}^{\infty} u^{(\mu)}(x)\phi^{(i)}(x)dx.$$

The ensemble average is defined as before, namely

$$\langle u(x) \rangle = \frac{1}{P} \sum_{\mu=1}^{P} u^{(\mu)}(x)$$

If the functions are time dependent, i.e., they are of the form $u(x,t)$ there is a related time-average defined as

$$\langle u(x) \rangle = \frac{1}{T} \int_0^T u(x,t)dt.$$

If the time-dependent function is sampled discretely in time we may collect an ensemble as before where

$$u^{(\mu)}(x) = u(x, t_\mu).$$

Often the approximation is made that the ensemble average and the time-average are equal. If this is true the flow is said to be *ergodic* [68].

As before, we proceed by defining the first basis function $\phi^{(1)}(x)$ by means of an optimization criterion. The mean-square projection of the data onto this function should be a maximum:

$$\langle (\int_{-\infty}^{\infty} u(x)\phi^{(1)}(x)dx)^2 \rangle = \text{maximum}$$

subject to

$$\int_{-\infty}^{\infty} (\phi^{(1)}(x))^2 dx = 1.$$

The remaining basis functions may be defined proceeding sequentially

$$\max_{\phi^{(j)}} \langle (\phi^{(j)}, u)^2 \rangle$$

subject to the side constraints

$$(\phi^{(j)}, \phi^{(k)}) = \delta_{jk}, \text{ for } k < j$$

Using the technique of Lagrange multipliers and the calculus of variations it can then be shown that the basis functions are solutions to the integral equation

$$\int C(x,y)\phi(y)dy = \lambda\phi(x) \tag{4.2}$$

where

$$C(x,y) = \langle u(x)u(y)\rangle \tag{4.3}$$

The type of this integral equation is known as a Fredholm Equation of the 2nd Kind and its properties fall within the scope of Hilbert-Schmidt theory [31]. This integral equation is seen to be the continuous analogue to the eigenvector problem for obtaining a best basis in a finite-dimensional setting. Among the properties of the solutions of this *eigenfunction* problem are the following:

- The kernel is symmetric, i.e., $C(x,y) = C(y,x)$.
- The eigenvalues are real, countable, and non-negative.
- The total energy is finite, i.e., $\sum_n \lambda_n < \infty$.
- The eigenfunctions form an orthonormal basis for the Hilbert space L^2.

In addition, we have the continuous analogue to the spectral theorem

Theorem 4.1. *Given* $C(x,y) = \langle u(x)u(y)\rangle$ *is continuous in* x, y *then*

$$C(x,y) = \sum_i \lambda_i \phi^{(i)}(x)\phi^{(i)}(y)$$

where the series converges uniformly and absolutely to $C(x,y)$.

This is known as Mercer's theorem. We refer the reader to [72, 65] for further details.

Returning to the data analysis problem, it is possible that the relative frequency $p^{(\alpha)}$ of each member of the ensemble is known, or may be estimated. If the ensemble consists of P elements, then $Pp^{(\alpha)}$ of the elements are indexed by α. The relative frequency satisfies the conditions

$$p^{(\alpha)} \geq 0, \quad \sum_{\alpha=1}^{P} p^{(\alpha)} = 1$$

The *weighted* covariance matrix is then defined as

$$C(x,y) = \sum_{\alpha=1}^{P} p^{(\alpha)} u(x)u(y) \tag{4.4}$$

This weighted covariance matrix may then used in place of the evenly weighted ensemble average covariance matrix.

In the next Section we see how the continuous transform allows us to extend these ideas to vector functions. Furthermore, we show that discrete data made up of concatenated vectors derived from continuous functions can be dealt with in a computationally efficient manner.

4.2 VECTOR FUNCTION KL EXPANSIONS

One of the most important applications of the KL expansion is to data consisting of several variables with values defined over a mulit-dimensional domain. This section deals with the extension of the continuous KL expansion to such vector functions. The procedure is also extended to the fully discrete setting.

4.2.1 Continuous Vector Functions

Here we address the application of the KL procedure to an ensemble of vector functions of the form

$$\mathbf{u}^{(\mu)}(\mathbf{x}) = (u_1^{(\mu)}(\mathbf{x}), \ldots, u_K^{(\mu)}(\mathbf{x}))^T$$

where each

$$u_i : \mathbb{R}^j \to \mathbb{R}$$

and j is the dimension of the domain. Such "data sets" are difficult to generate in practice, but they arise naturally in theoretical settings.

Example 4.1. Consider the fluid flow with scalar flow variables consisting of the concatenated vector function

$$\mathbf{u}(\mathbf{x}, t) = (u(\mathbf{x}, t), v(\mathbf{x}, t), w(\mathbf{x}, t), e(\mathbf{x}, t), \rho(\mathbf{x}, t))^T$$

where $\mathbf{x} = (x_1, x_2, x_3)$ and u, v, w are the flow velocities in the x_1, x_2 and x_3 directions, respectively; e is the internal energy and ρ is the fluid density.

To determine a best basis for such vector functions we must extend the definition of the kernel of the integral equation. The appropriate extension is

$$\mathbf{C}(\mathbf{x}, \mathbf{x}') = \langle \mathbf{u}(\mathbf{x})\mathbf{u}(\mathbf{x}') \rangle.$$

where

$$\mathbf{C}_{ij}(\mathbf{x}, \mathbf{x}') = \langle u_i(\mathbf{x}) u_j(\mathbf{x}') \rangle.$$

The kernel $\mathbf{C}$ is now referred to as a two-point correlation tensor. If $K = 2$ we see that

$$\begin{pmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} \\ \mathbf{C}_{21} & \mathbf{C}_{22} \end{pmatrix} = \begin{pmatrix} \langle u_1(\mathbf{x})u_1(\mathbf{x}') \rangle & \langle u_1(\mathbf{x})u_2(\mathbf{x}') \rangle \\ \langle u_2(\mathbf{x})u_1(\mathbf{x}') \rangle & \langle u_2(\mathbf{x})u_2(\mathbf{x}') \rangle \end{pmatrix}$$

The integral equation which produces the optimal eigenfunctions is now given by

$$\int \mathbf{C}(\mathbf{x}, \mathbf{x}')\boldsymbol{\phi}(\mathbf{x}')\mathbf{dx}' = \lambda\boldsymbol{\phi}(\mathbf{x}).$$

This may be thought of $\boldsymbol{\phi}(\mathbf{x})$ as being a concatenation of components of eigenfunctions

$$\boldsymbol{\phi}(\mathbf{x}) = (\phi_1(\mathbf{x}), \phi_2(\mathbf{x})).$$

The integral equation can also be written in component form as

$$\int \sum_{j=1}^{2} \mathbf{C}_{ij}(\mathbf{x}, \mathbf{x}')\phi_j(\mathbf{x}')\mathbf{dx}' = \lambda\phi_i(\mathbf{x}).$$

for $i = 1, 2$.

The Snapshot Method

The snapshot method again helps us reduce the problem for degenerate kernels. We again start with the data-dependent representation

$$\boldsymbol{\phi}(\mathbf{x}) = \sum_{\mu=1}^{P} a_\mu \mathbf{u}^{(\mu)}(\mathbf{x})$$

and component-wise

$$\phi_j(\mathbf{x}) = \sum_{\mu=1}^{P} a_\mu u_j^{(\mu)}(\mathbf{x}).$$

Substituting this into the component integral equations leads to

$$\int \sum_j \langle u_i(\mathbf{x}) u_j(\mathbf{x}')\rangle (\sum_\mu a_\mu u_j^{(\mu)}(\mathbf{x}'))\mathbf{dx}' = \lambda \sum_\nu a_\nu u_i^{(\nu)}(\mathbf{x}).$$

Expanding the ensemble average

$$\int \sum_j (\frac{1}{P} \sum_\nu u_i^{(\nu)}(\mathbf{x}) u_j^{(\nu)}(\mathbf{x}'))(\sum_\mu a_\mu u_j^{(\mu)}(\mathbf{x}'))\mathbf{dx}' = \lambda \sum_\nu a_\nu u_i^{(\nu)}(\mathbf{x}).$$

Rearranging,

$$\sum_\nu u_i^{(\nu)}(\mathbf{x})[\sum_\mu a_\mu \sum_j \int u_j^{(\nu)}(\mathbf{x}') u_j^{(\mu)}(\mathbf{x}')\mathbf{dx}' - (\lambda P)a_\nu] = 0.$$

In other words,

$$\sum_\nu u_i^{(\nu)}(\mathbf{x})[\sum_\mu \mathbf{L}_{\nu\mu} a_\mu - \lambda P a_\nu] = 0$$

where

$$\mathbf{L}_{\nu\mu} = \int \sum_j u_j^{(\nu)}(\mathbf{x}') u_j^{(\mu)}(\mathbf{x}') \mathbf{dx}'.$$

Hence,

$$\mathbf{La} = \tilde{\lambda}\mathbf{a}$$

where we have used $\tilde{\lambda} = \lambda P$.

4.2.2 Discrete Vector Functions

As we have already observed, pattern data typically comes not in the form of continuous functions but rather as discrete vectors. The discussion in Section 3.4 dealt directly with the application of the KL procedure to the case of a single independent discrete variable. The results of the previous Section apply to continuous vector functions. We now modify these results so that we can treat discretely sampled vector data.

Example 4.2. A typical source of discrete multivariable vector data is a numerical simulation of physical problem, such as the motion of a fluid. The continuous data described in Example 4.1 is now estimated on a 3-dimensional lattice, or grid. For example, the first component of velocity $u(\mathbf{x}, t)$ is computed on $u(x_1, x_2, x_3, t)$ where $x_i \in \{1, \ldots, N_i\}$. This data may be concatenated into a single column vector as

$$\mathbf{u}(t) = \begin{pmatrix} u(1,1,1) \\ \vdots \\ u(N_1, N_2, N_3) \end{pmatrix}$$

Example 4.3. A color digital image is commonly given as a triplet of red, $r(i,j)$, green, $g(i,j)$ and blue $b(i,j)$ color values on a two-dimensional lattice. Concatenating the rows (or columns) of each image, and then the images leads to the high-dimensional vector

$$\mathbf{x} = (\mathbf{r}, \mathbf{g}, \mathbf{b})^T.$$

Lets consider the application of the above where the continuous vector function is actually discretized so that

$$\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2)^T$$

where

$$\mathbf{u}_1, \mathbf{u}_2 \in \mathbb{R}^n \quad \mathbf{u} \in \mathbb{R}^{2n}.$$

Then

$$\begin{aligned} \mathbf{C} &= \langle \mathbf{u}\mathbf{u}^T \rangle \\ &= \begin{pmatrix} \langle \mathbf{u}_1\mathbf{u}_1^T \rangle & \langle \mathbf{u}_1\mathbf{u}_2^T \rangle \\ \langle \mathbf{u}_2\mathbf{u}_1^T \rangle & \langle \mathbf{u}_2\mathbf{u}_2^T \rangle \end{pmatrix} \end{aligned}$$

Thus each $\mathbf{C}_{ij}$ is an $n \times n$ matrix and $\mathbf{C}$ is a $2n \times 2n$ matrix. Again we have the eigenvector problem

$$\mathbf{C}\phi = \lambda\phi$$

which leads to the snapshot equation, as in the continuous case considered in the previous Section, i.e.,

$$\mathbf{La} = \lambda P\mathbf{a}$$

where $\mathbf{L}$ is a $P \times P$ matrix and $\mathbf{L}_{\nu\mu} = \sum_j (\mathbf{u}_j^{(\nu)}, \mathbf{u}_j^{(\mu)})$, i.e., a vector dot product.

4.3 SYMMETRIC OPTIMAL EIGENFUNCTIONS

Historically, one of the most important eigenfunction expansion is the continuous Fourier series

$$f(x) = \sum_k a_k \cos(kx) + b_k \sin(kx) \tag{4.5}$$

which decomposes a periodic function into its even and odd parts. The $\{a_k\}$ coefficients represent the odd portion of $f(x)$ while the $\{b_k\}$ represent the even. It is easy to show using the orthogonality of the sinusiods that if $f(x)$ is even, then the $\{b_k\}$ are all zero, and if $f(x)$ is odd, then the $\{a_k\}$ are all zero. This property often simplifies the computations associated with using the Fourier series expansion.

Recall that a function $f(x)$ is said to be even if $f(x) = f(-x)$ and odd if $f(x) = -f(-x)$. In addition, any function may be expressed as the sum of an even and odd function using the identity

$$f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2}.$$

For example,

$$e^x = \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2}$$

See Figure 4.1 for the plots of the individual functions in the decomposition.

The decomposition of a function into even and odd orthogonal components may be generalized to the form

$$f(x) = \sum_k a_k f_e^{(k)}(x) + b_k f_o^{(k)}(x) \tag{4.6}$$

where $\{f_e^{(k)}(x)\}$ are even functions, $\{f_o^{(k)}(x)\}$ are odd functions, and together they form a basis for the function space in question. Given such a decomposition it is possible to characterize the symmetry of a pattern or an ensemble of patterns. For instance, the degree of *evenness* of an ensemble may be quantified by the sum $\sum_k \langle a_k^2 \rangle$ and degree of *oddness* as $\sum_k \langle b_k^2 \rangle$. The symmetric KL procedure will automatically produce these quantities as eigenvalues.

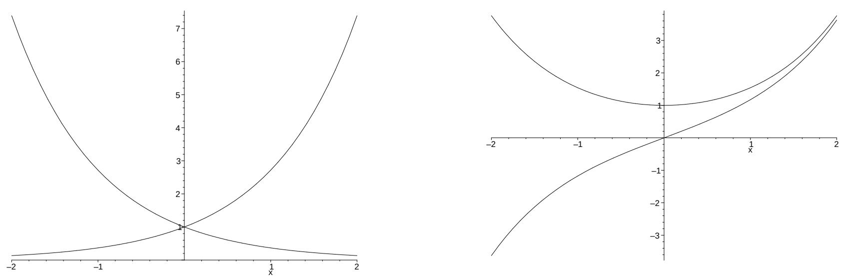

Fig. 4.1 Left: the function e^x and its reflection e^{-x}. Right: The functions on the left may be decomposed in terms of the even and odd functions $f_e(x) = (e^x + e^{-x})/2$, $f_o(x) = (e^x - e^{-x})/2$.

Given the convenience of such a decomposition for characterizing the symmetric components of a function (or data vector) it is natural to address the issue of symmetric optimal eigenfunction (or eigenvector) expansions.

In general, if $C(x, y) = \langle u(x)u(y)\rangle$ and the optimal eigenfunctions in the expansion are determined by solving the equation

$$\int C(x,y)\phi(y)dy = \lambda\phi(x) \tag{4.7}$$

then the eigenfunctions $\{\phi^{(j)}\}$ are neither even nor odd, i.e., the possess no symmetry. As a consequence, the optimal eigenfunction expansion

$$u(x) = \sum_{j=1}^{\infty} a_j \phi^{(j)}(x)$$

does not permit the splitting of the decomposition into even and odd subspaces.

The topic of this Section is to demonstrate how a simple modification to the KL procedure permits the construction of optimal bases with even and odd eigenfunctions. Although the initial setting for the discussion is the continuous transform, all of the ideas carry over to the discrete case.

We begin by defining the reflection operator

$$R\phi(x) = \phi(-x).$$

Now consider an ensemble of patterns $\{u^{(\mu)}(x)\}$ with $\mu = 1, \ldots, P$.

Definition 4.1. *The symmetry extended ensemble is defined as the union*

$$\tilde{X} = \{u^{(\mu)}(x)\} \cup \{Ru^{(\mu)}(x)\}$$

where $\mu = 1, \ldots, P$.

The extended ensemble average is then given by

$$\langle u(x) \rangle = \frac{1}{2P} \sum_{\mu=1}^{P} (u^{(\mu)}(x) + Ru^{(\mu)}(x)).$$

It is interesting to note that the extended ensemble average is an even function. As usual, we form the fluctuating ensemble

$$\tilde{u}^{(\mu)}(x) = u^{(\mu)}(x) - \langle u(x) \rangle$$

and for simplicity we again immediately drop the tilde notation.

It will be demonstrated that the symmetric optimal eigenfunction expansion is found by solving the *symmetrized* integral equation

$$\int \hat{C}(x,y)\hat{\phi}(y)dy = \lambda\hat{\phi}(x) \tag{4.8}$$

where the symmetrized kernel $\hat{C}$ is given by

$$\hat{C}(x,y) = \frac{1}{2P} \sum_{\mu=1}^{P} (u^{(\mu)}(x)u^{(\mu)}(y) + Ru^{(\mu)}(x)Ru^{(\mu)}(y)). \tag{4.9}$$

Now we propose to compare the solutions of the symmetrized integral Equation (4.8) with those of the original eigenfunction problem of Equation (4.7). In particular, it will be shown that the eigenfunctions of Equation (4.8) are symmetric, i.e., even and odd functions. In addition, it will be shown that the solutions of the symmetrized integral equation may be found by solving associated even and odd integral equations.

The set of eigenfunctions which satisfy a given integral equation also define an eigenspace, in a manner analogous with the discrete eigenvector problem. For example, the eigenspace associated with the symmetrized kernel $\hat{C}$ is the set of functions

$$E(\hat{C}) = \{\hat{\phi} : \int \hat{C}(x,y)\hat{\phi}(y)dy = \lambda\hat{\phi}(x)\} \tag{4.10}$$

The analysis of Equation (4.8) will be facilitated by the introduction of the even functions

$$u_e^{(\mu)}(x) = \frac{u^{(\mu)}(x) + Ru^{(\mu)}(x)}{2}$$

and the associated odd functions

$$u_o^{(\mu)}(x) = \frac{u^{(\mu)}(x) - Ru^{(\mu)}(x)}{2}.$$

For $\mu = 1, \ldots, P$ we may view these functions as defining an even ensemble $\{u_e^{(\mu)}(x)\}$ and an odd ensemble $\{u_o^{(\mu)}(x)\}$.

These ensembles also define kernels. The even kernel

$$C_e(x,y) = \langle u_e(x)u_e(y)\rangle$$

is even in x and y and the odd kernel

$$C_o(x,y) = \langle u_o(x)u_o(y)\rangle$$

is odd in x and y.

Proposition 4.1. *The symmetrized kernel $\hat{C}$ may be decomposed into even and odd components using these kernels, i.e.,*

$$\hat{C}(x,y) = C_e(x,y) + C_o(x,y)$$

This splitting of the kernel into its even and odd components will permit us to establish that the integral equations which produce the even and odd eigenfunctions are the *even* integral equation

$$\int C_e(x,y)\phi_e(y)dy = \lambda\phi_e(x) \tag{4.11}$$

and the *odd* integral equation

$$\int C_o(x,y)\phi_o(y)dy = \lambda\phi_o(x) \tag{4.12}$$

respectively.

Theorem 4.2. *If $\hat{\phi}$ is an eigenfunction of the symmetrized kernel $\hat{C}$, i.e.,$\hat{\phi} \in E(\hat{C})$ and $\hat{\phi}(x) = \hat{\phi}_e(x) + \hat{\phi}_o(x)$ decomposes $\hat{\phi}$ into even and odd functions, then*

- $\hat{\phi}_e(x) \in E(C_e)$
- $\hat{\phi}_o(x) \in E(C_o)$

In follows that,

$$E(\hat{C}) \subseteq E(C_e) + E(C_o) \tag{4.13}$$

Proof. By assumption

$$\int \hat{C}(x,y)\hat{\phi}(y)dy = \lambda\hat{\phi}(x)$$

Decomposing into the even and odd components gives

$$\int (C_e(x,y) + C_o(x,y))(\hat{\phi}_e(y) + \hat{\phi}_o(y))dy = \lambda(\hat{\phi}_e(x) + \hat{\phi}_o(x))$$

After multiplying out and setting the appropriate terms to zero (see Exercise 4.7) we have

$$\int C_e(x,y)\hat{\phi}_e(y)dy + \int C_o(x,y)\hat{\phi}_o(y)dy = \lambda(\hat{\phi}_e(x) + \hat{\phi}_o(x))$$

Equating the even and odd parts produces Equations (4.11) and (4.12), respectively. □

The next theorem essentially says that the even and odd eigenfunctions are actually solutions to the symmetrized integral equation. In other words, $E(C_e) \subset E(\hat{C})$ and $E(C_o) \subset E(\hat{C})$.

Theorem 4.3. *If $\phi_e \in E(C_e)$, then $\phi_e \in E(\hat{C})$. Similarly, if $\phi_o \in E(C_o)$, then $\phi_o \in E(\hat{C})$.*

Proof. By assumption

$$\int C_e(x,y)\phi_e(y)dy = \lambda\phi_e(x)$$

So $\int (C_e(x,y) + C_o(x,y))\phi_e(y)dy = \lambda\phi_e(x)$ since the additional term is just zero. Hence it follows

$$\int \hat{C}(x,y)\phi_e(y)dy = \lambda\phi_e(x)$$

The proof for ϕ_o is analogous. □

From these results we may conclude that any solution ϕ_e of (4.11) or ϕ_o of (4.12) is a solution to (4.8) and that the even and odd eigfunctions span orthogonal subspaces. Furthermore, it also follows that

$$E(C_e) + E(C_o) \subseteq E(\hat{C})$$

Thus, by combining the statements of theorems 4.13 and 4.13, we conclude that

$$E(C_e) + E(C_o) = E(\hat{C})$$

It is also easy to show that the even and odd kernels produce orthogonal eigenfunctions.

Proposition 4.2. *The even and odd eigenfunctions are orthgonal, i.e.,*

$$E(C_e) \perp E(C_o)$$

The proof of this is left for the exercises.

In other words $E(\hat{C})$, the eigenspace determined by equation (4.8) can be expressed as the direct sum of $E(C_e)$ and $E(C_o)$; i.e.,

$$E(\hat{C}) = E(C_e) \oplus E(C_o).$$

Thus to determine the even eigenfunctions $\{\hat{\phi}_e(y)\}$, and the odd eigenfunctions $\{\hat{\phi}_o(y)\}$ of $\hat{C}$, we may solve the "smaller" problems given by Equations (4.11) and (4.12).

This theorem shows that the eigenfunctions of the symmetrized integral Equation (4.8) are actually all either even or odd.

Theorem 4.4. *Assume that all eigenvalues of the symmetrized integral equation are distinct. If $\hat{\phi}$ is an eigenfunction of $\hat{C}$, then either $\hat{\phi} = \hat{\phi}_e$, or $\hat{\phi} = \hat{\phi}_o$.*

Proof. Since $\hat{\phi}$ is an eigenvector of $\hat{C}$ and $\hat{\phi} = \hat{\phi}_e + \hat{\phi}_o$ it follows

$$\int \hat{C}(x,y)(\hat{\phi}_e(y) + \hat{\phi}_o(y))dy = \lambda(\hat{\phi}_e(x) + \hat{\phi}_o(x)) \tag{4.14}$$

Also, by theorems 4.2 and 4.3, these eigenfunctions satisfy $\int \hat{C}(x,y)\hat{\phi}_e(y)dy = \lambda_e\hat{\phi}_e(x)$ and $\int \hat{C}(x,y)\hat{\phi}_o(y)dy = \lambda_o\hat{\phi}_o(x)$, or, after adding these equations together,

$$\int \hat{C}(x,y)(\hat{\phi}_e(y) + \hat{\phi}_o(y))dy = \lambda_e\hat{\phi}_e(x) + \lambda_o\hat{\phi}_o(x) \tag{4.15}$$

Equating the right-hand sides of Equations (4.15) and (4.14) leads to

$$\lambda(\hat{\phi}_e(x) + \hat{\phi}_o(x)) = \lambda_e\hat{\phi}_e(x) + \lambda_o\hat{\phi}_o(x)$$

or

$$(\lambda - \lambda_e)\hat{\phi}_e(x) + (\lambda_o - \lambda)\hat{\phi}_o(x) = 0$$

But $\hat{\phi}_e(x)$ and $\hat{\phi}_o(x)$ are independent so $\lambda = \lambda_e = \lambda_o$. But this is a contradiction, since the eigenvalues are distinct by assumption. The only remaining possibilities are

- $\hat{\phi}_e(x) = 0$, $\lambda = \lambda_o$, $\hat{\phi}(x) = \hat{\phi}_o(x)$
- $\hat{\phi}_o(x) = 0$, $\lambda = \lambda_e$, $\hat{\phi}(x) = \hat{\phi}_e(x)$

□

4.3.1 Symmetric Optimal Eigenvectors

Although the continuous KL transform was a useful setting to derive the symmetric properties of the eigenfunctions for a symmetry extended data set, the discrete formulation is used in practical computations. This section

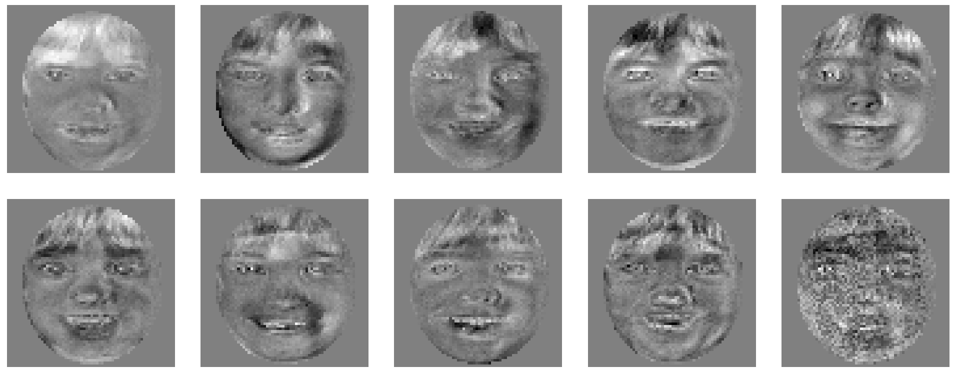

Fig. 4.2 The eigenvectors of a mean-subtracted ensemble of 10 faces.

outlines the discrete procedure which is analogous to the continuous KL of the previous section.

Definition 4.2. *A vector* $\mathbf{x} \in \mathbb{R}^N$ *is said to be even if*

$$x_i = x_{N-i+1}$$

and odd if

$$x_i = -x_{N-i+1}$$

for $i = 1, \ldots, N$.

Following the notation of the previous section we define the reflection of a vector about its midpoint as

$$(R\mathbf{x})_i = x_{N-i+1} \tag{4.16}$$

Example 4.4. A vector of length 4 may be said to be even if it has the form (a, b, b, a) and odd if it has the form $(a, b, -b, -a)$. All vectors of length 4 may be decomposed into the sum of an even and odd vector using

$$\mathbf{x} = \frac{1}{2}\begin{pmatrix} x_1 + x_4 \\ x_2 + x_3 \\ x_2 + x_3 \\ x_1 + x_4 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} x_1 - x_4 \\ x_2 - x_3 \\ -x_2 + x_3 \\ -x_1 + x_4 \end{pmatrix} \tag{4.17}$$

Following the previous section, the symmetric eigenvectors are computed from the symmetrized eigenvector problem

$$\hat{C}\hat{\phi} = \lambda\hat{\phi} \tag{4.18}$$

where

$$\hat{C} = \frac{\langle \mathbf{u}\mathbf{u}^T \rangle + \langle R\mathbf{u}R\mathbf{u}^T \rangle}{2}$$

There are twice as many patterns in this ensemble, so for rank deficient problems it is computationally less expensive to solve the associated even and odd eigenvector problems

$$C_e \boldsymbol{\phi}_e = \lambda \boldsymbol{\phi}_e$$

and

$$C_o \boldsymbol{\phi}_o = \lambda \boldsymbol{\phi}_o$$

where C_e and C_o are defined as before. Again, the even and odd patterns are found using

$$\mathbf{x}_e = \frac{\mathbf{x} + R\mathbf{x}}{2} \quad \mathbf{x}_o = \frac{\mathbf{x} - R\mathbf{x}}{2}$$

where the action of R is defined by Equation (4.16).

The Snapshot Method

When the matrices C_o and C_e are singular it is again useful to employ the Snapshot Method. Specifically, representing the even and odd eigenvectors in terms of the even and odd data we have

$$\boldsymbol{\phi}_o = \sum_\mu b_\mu \mathbf{x}_o^{(\mu)}, \quad \boldsymbol{\phi}_e = \sum_\mu a_\mu \mathbf{x}_e^{(\mu)}$$

yields the two reduced problems

$$L^e \mathbf{a}^{(\mu)} = \lambda \mathbf{a}^{(\nu)},$$

$$L^o \mathbf{b}^{(\mu)} = \lambda \mathbf{b}^{(\nu)}$$

where $L^e_{\nu\mu} = (\mathbf{x}_e^{(\nu)}, \mathbf{x}_e^{(\mu)})$ and $L^o_{\nu\mu} = (\mathbf{x}_o^{(\nu)}, \mathbf{x}_o^{(\mu)})$. Now there are two $P \times P$ eigenvector problems rather than a single $2P \times 2P$ problem which arises if we solve Equation (4.18) via the Snapshot method. Again, the resolution of the patterns is a factor only in the computation of the dot products and the required memory. As a result, very high-resoultion images may present a practical problem even if the eigenvector problems can be solved.

Example 4.5. *The Symmetric Rogues Gallery Problem.* The representation of digital images in Section 3.6.1 may now be extended to include symmetry. As an example, we revisit the Rogues Gallery problem and compute the even and odd eigenpictures. The result of computing the eigenpictures of the even ensemble is shown at the top of Figure 4.3; as expected, they are all even about the midline. The bottom of Figure 4.3 shows the result of computing the eigenpictures of the odd ensemble. Again, as expected, they are odd about the midline. For purposes of comparison the eigenpictures for the unextended ensemble are shown in Figure 4.2. See [44] for further details.

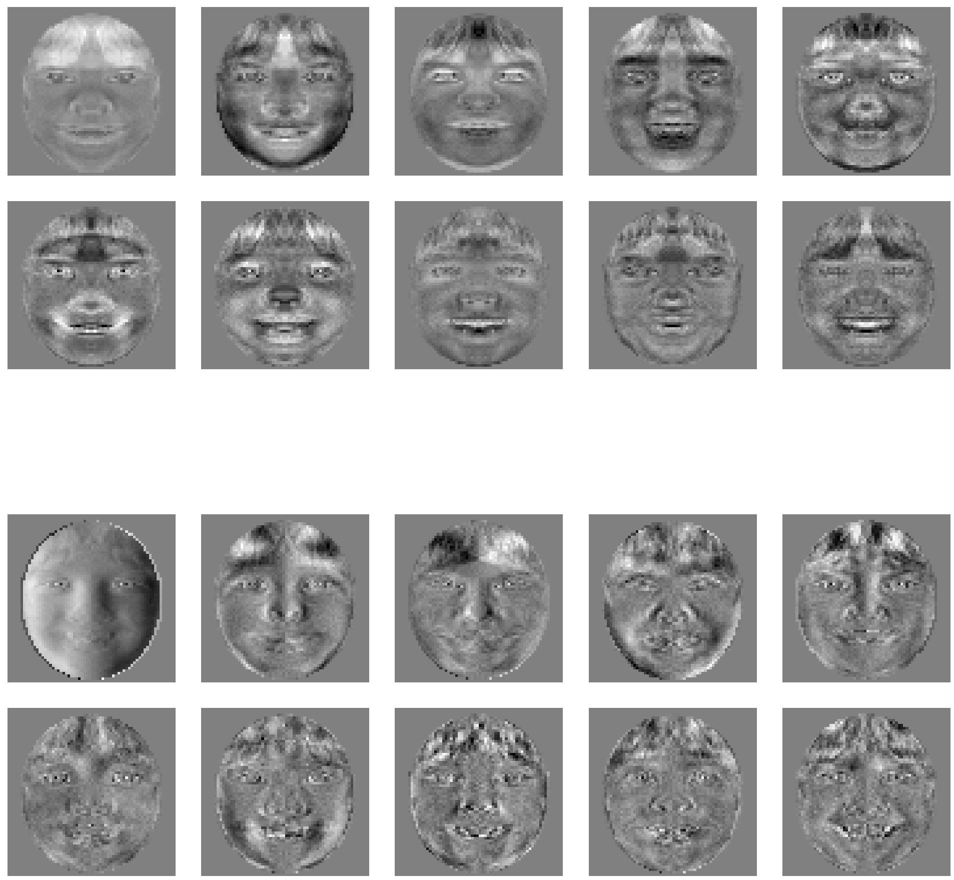

Fig. 4.3 Top: the even eigenvectors. Bottom: the odd eigenvectors.