M435 Projects in Applied Mathematics
Singular Value Decomposition Project

This sample project explores the application of the singular value decomposition to image analysis. If your final project is limited to this sample then please hand in your work individually. As a group you are expected to expand upon this project, possibly using another data set or an additional algorithm.

The application concerns the low-rank approximation of a digital image of the planet Saturn. This image will be loaded into a data matrix X.

1 Theory

1. Show that the non-zero singular values of the matrix X are the same as the positive square roots of the eigenvalues of XX^T and X^TX.

2. Describe the meaning of a plot of the eigenvalues of XX^T in decreasing order, i.e., largest to smallest.

3. Explain the meaning of zero singular values. What is the largest number of non-zero singular values possible for a $P \times N$ data matrix X? (Consider the decomposition $X = UDV^T$.

4. Show that the left (and right) singular vectors may be viewed as data-dependent by expressing them as sums of the columns (rows) of the data matrix X.

5. Pose and answer a questions about the SVD.

2 Computing

The Saturn image for this problem may be downloaded from the class web page. The format of the data is "uint8", which stands for unsigned integer, 8 bits. Before you apply the SVD, change it to "double" format:

```
A=imread('saturn.tif')
A=double(A);
```

To display the image use

```
imagesc(A);
colormap(gray);
```

Use the singular value decomposition to construct a low-rank approximation to Saturn.

```
[U,D,V] = svd(X);
```

1. Plot the squares of the singular values in decreasing order. Based on this graph can you comment on how many terms you might expect to need to accurately reconstruct Saturn?

2. Recalculate the SVD of the data matrix of Saturn but first subtract the ensemble (row) average. Compare your singular values and left singular vectors with the previous calculation. Examine the ratio of singular values and comment.

3. Plot the rank one images for several different eigenvalues (large, medium, and small) and compare.

4. You may observe that the rings of Saturn are poorly approximated at the edges. Plot the 4 left singular vectors that are most important (based on maximum mean squared projection) for "fitting" the rings. Repeat for the right singular vectors and comment.

5. Can you modify the ordering of your basis to better approximate the rings? Show your results.

6. Write a subroutine to automatically identify whether a column of the picture contains only rings or both planet and rings.