Mathematical Modeling, Proof of Concept

Mathematical Modeling

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CHAPTER 7

Simulation Modeling

It is not unusual that the complexity of a phenomenon or system makes a direct mathematical attack time-consuming, or worse, intractable. An alternative modeling approach consists of the literal execution of rules by the computer. Such *simulation* approaches occur in great variety but share the common feature that a computer is the central vehicle for knowledge, or process discovery. Here are some problems where simulation modeling appears to be successful, if not indispensible:

- Develop strategies in games with simple rules but stochastic components such as blackjack, checkers, and solitaire.
- Numerically approximate solutions to complex models such as the Navier-Stokes equations of fluid dynamics. (These simulations are typically deterministic).
- Modeling phenomenon with inherent probabilistic components such as processing queues, traffic problems, and inventory problems.

This above list, while certainly not complete, gives an indication of the range of problems that lend themselves naturally to simulation modeling.

7.1. The Tire Distributor Problem

Consider the inventory problem confronting the distributor of a commodity with random demand. To be concrete we will consider the situation of a car tire distributor. Our assumption is that this distributor supplies tires to a large number of clients and in turn purchases its supply of tires from a major tire producer. The distributor itself does not fabricate tires but relies on deliveries from the factory. Based on observed daily demand it is up to the distributor to determine a quantity X of tires to be delivered at an integer interval N in days.

The problem is driven by costs. It is the desire of the distributor to select an X and N to minimize total costs. These costs are assumed to consist of two components:

- tire delivery costs
- interest costs on money borrowed to pay for tires in stock

The tire delivery costs may be modeled in several different ways. We will assume that the factory delivers and charges by the truckload and each truckload can contain up to 1,000 tires. Furthermore, the delivery charges for a truckload with no discount if the truck is not full. Hence, if a truckload costs $\$\alpha$, then the delivery costs are $d(X)$ are

$$d(X) = \alpha\left([\frac{X-1}{1000}] + 1 \right)$$

where the notation $[a]$ means the the number a rounded down to the nearest integer. So, for example,

$$\left[\frac{1500}{1000}\right] = 1$$

The second cost is due to the interest charged by the bank for the load the company has to purchase the tires. Note that even if the distributor had enough cash flow to not need to borrow, we would still view the investment in the tires sitting in the warehouse as a cost as this money could presumably be invested to earn a dividend.

Let's assume the interest rate per day per tire is β. If x_n tires are in stock at the end of day n then we are assessed the interest fee βx_n so that the total cost of interest over an interval N is

$$I = \sum_{n=1}^{N} \beta x_n$$

This problem is complicated by the fact that the number of tires purchased by customers varies randomly from day to day as shown in Table 1. Furthermore, the average demand was calculated to be 997 tires/day. Thus, the distributor must attempt to simulate this statistical demand in the model. To accomplish this we need to develop a demand subroutine that will mimic the observed daily demand. This may be achieved by associating the demand intervals with segments of the unit interval the length of which is determined by the actual frequence of demand. So, for example, the demand interval $0 \leq x_n < 100$ occurs 12 days out of 365 (presumably due to holidays). So, we associate the interval

$$I_1 = [0, 12/365]$$

to the probability that the daily demand will be in the interval $0 \leq x_n < 100$ tires. Since we are mapping the interval [0,1] to daily demand ranges we must require that these subintervals be nonoverlapping. Thus, the fact that the interval $100 \leq x_n < 300$ occurred on exactly 4 days means that we should reserve 4/365 of the unit interval for this demand range, i.e., $I_2 = [12/365, 16/365]$. We may develop the rest of the intervals in a similar fashion, i.e.,

$$I_3 = [16/365, 43/365]$$

$$I_4 = [43/365, 86/365]$$

and so on.

Now we have partitioned the unit interval, i.e.,

$$[0, 1] = I_1 \cup I_2 \cup \cdots \cup I_{11}$$

Thus, if we pick a random number $z \in [0, 1]$ (always picking this number uniformly from the interval) we can map that number to an appropriate daily demand interval.

For example, if our uniform random number generator returns $z = .0768$ then we select the interval containing this point, i.e., I_3. Now the question remains how to pick an actual daily demand? If we are in interval I_3 we only know that the demand should be between 300 and 500 tires. So we may select this demand randomly from the integers in this interval. This model we have constructed for simulating the demand can be simply tested by running the model for 100 years and seeing if we reproduce the demands (now averaged over the 100 years). The results of doing this, shown in the last column of table 1 suggest this model is rather good.

daily demand	frequency in days	cumulative distribution	simulated freq.
$0 \leq x_n < 100$	12	12	12.22
$100 \leq x_n < 300$	4	16	4.04
$300 \leq x_n < 500$	27	43	26.67
$500 \leq x_n < 700$	43	86	43.05
$700 \leq x_n < 900$	48	134	47.20
$900 \leq x_n < 1100$	67	201	66.31
$1100 \leq x_n < 1300$	78	279	78.92
$1300 \leq x_n < 1500$	55	334	55.77
$1500 \leq x_n < 1700$	22	356	21.56
$1700 \leq x_n < 190$	7	363	7.24
$1900 \leq x_n < 2100$	2	365	2.02

TABLE 1. Number of days certain quantities of tires were demanded. Total number of days of collected data is 365.

Summary of Tire Distributor Simulation

We run the simulation for 365 days and compute the average daily cost. This calculation is repeated 100 times and the average cost is now averaged again. This produces a more reliable stochastic estimate of the cost.

- Make an initial delivery of tires of size deliver_quantity.
- Compute the stochastic demand and subtract sales from stock NUM_TIRES
- If the stock is zero then add a penalty
- Accrue interest costs every iteration.
- If delivery_interval counter indicates delivery then increment NUM_TIRES by delivery_quantity.

7.2. Blackjack Strategy

Blackjack is a poker game pitting the Dealer, or Bank, against one or more players. For simplicity we will assume there is only one player, as in video blackjack. The object of the game is to score higher than the dealer without going over 21 points. If this occurs the player is paid the value of his bet. If the Dealer has a higher score than the player the Dealer collects the bet. Ties result in no loss of bet, or a *push*.

Card Values. The 10, Jack, Queen, and King are all valued at 10 while the cards from 2 through 9 are valued as indicated. The score of a hand is obtained by adding the values of the individual cards in the hand. The Ace is worth 1 or 11 points. The player can choose the value of the Ace whereas the dealer must always take the high value unless the total is over 21.

Blackjack occurs when a total of 21 is obtained with the first two cards in a hand (a black Jack is not necessary). If the Dealer has Blackjack and the Player has 21, the Dealer wins. If the Dealer has 21 and the Player has Blackjack, the Player wins. If a Player wins with a Blackjack then he is paid 1.5 times the bet placed. A total of 21 with more than two cards is not black jack.

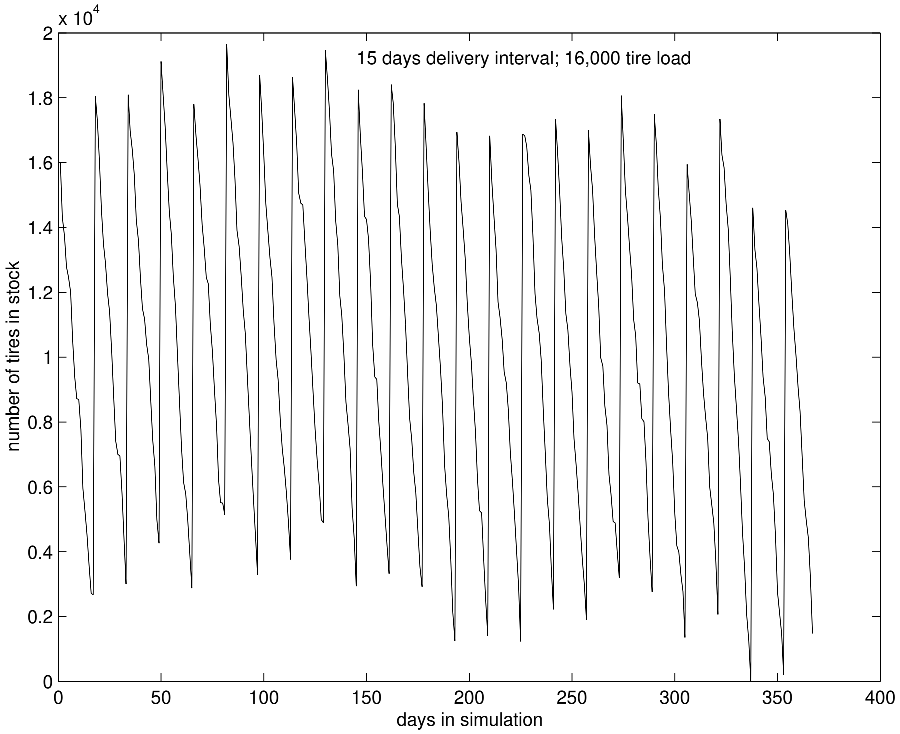

FIGURE 1. The average daily cost for this simulation averaged over 365 days was \$228. This simulation has no days without tires and does not tend to accumulate tires that would be subject to interest.

Rules of Play. The dealer distributes two cards to the Player and to himself. The Dealer shows the value of one of his cards to the player. The Player then requests cards one at a time until he decides to stay pat (receive no more cards) or goes bust (exceeds 21). If the Player stands pat then the Dealer plays. The Dealer has no choices in how he plays his cards but must follow a specific set of rules.

Dealer Play. Dealer must stand pat on 17. Dealer must continue to take cards until his total is 17 or higher An Ace in the dealer's hand is always counted as 11 unless counting it as 1 prevents the dealer from going over. Thus if the dealer holds (ace,7) then he must stand. If the dealer holds (10,6,Ace) where the Ace is the third card picked then the dealer holds 17 and must stand.

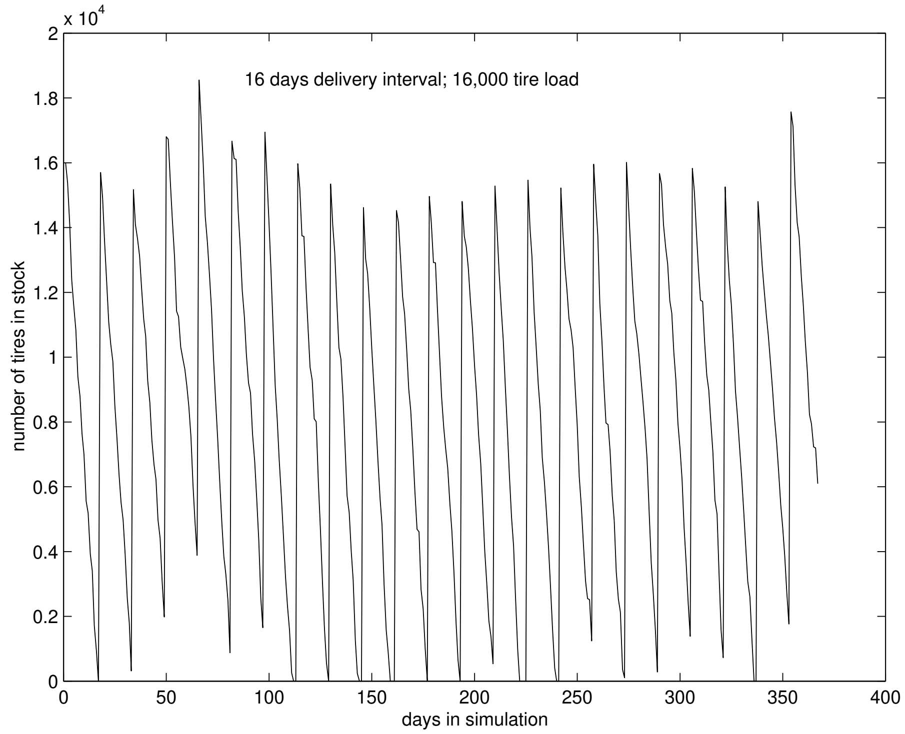

FIGURE 2. The average daily cost for this simulation averaged over 365 days was \$703. The increased cost is due to number of days (19) where the stock went to zero.

PROBLEM 7.1. The tire factory that supplies our distributor is overstocked and has decided to discount shipping such that the first truck costs the usual amount $\alpha = \$400$, the second truck costs $\$\alpha/2$, the third truck costs $\$\alpha/3$ and so on. Modify the tire distributor problem to account for this and determine new optimal values of delivery_amount and delivery_interval. Are your answers what you would expect?

PROBLEM 7.2. Modify the ordering of a new shipment of tires to occur when the stock is at 20% of the delivery load. How does this effect the optimal delivery amount and interval? Does the smallest average daily cost go down?

PROBLEM 7.3. Modify your code so that the number of tires in stock can't excede $T_{\max} = 17,000$ tires. Include graphs of your new simulations to demonstrate this limited capacity. What is the new optimal delivery_interval and delivery_quantity?

PROBLEM 7.4. Modify the ordering of a new shipment of tires to occur when the stock is at 20% of the delivery load. How does this effect the optimal delivery amount and interval? Does the smallest average daily cost go down?

PROBLEM 7.5. Now assume that the placement of orders for new shipments of tires have a random component based on the routines of the three rotating managers. Assume manager I works 30% of the shifts and that he will place the

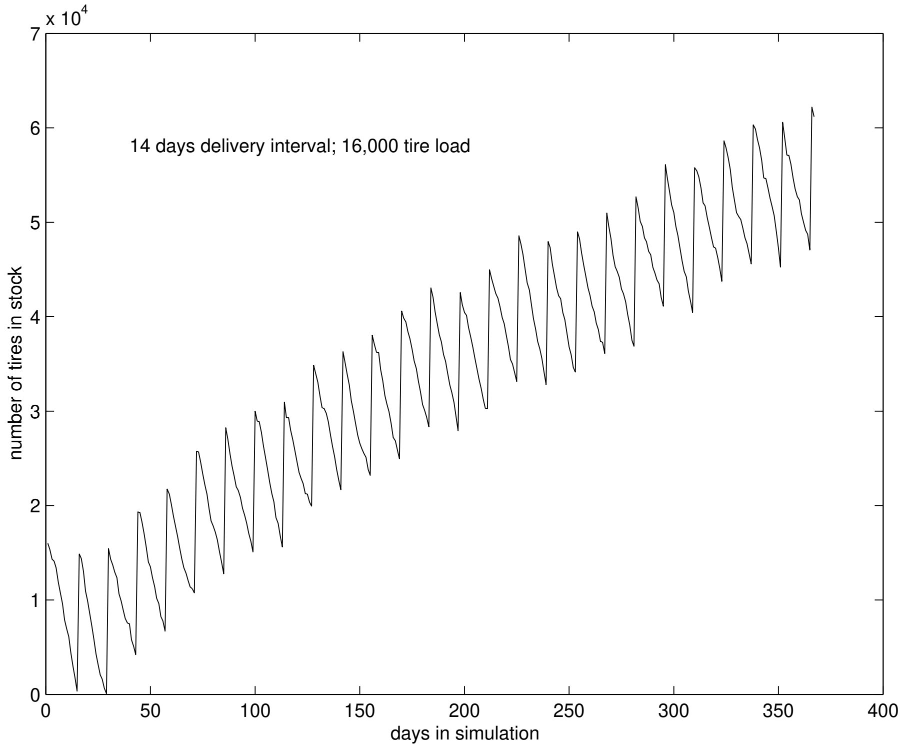

FIGURE 3. The daily cost for this simulation averaged over 365 days was \$447. This increased cost is apparently due to the interest costs on the increasing stock.

order when stock dips under 35% *of the delivery_quantity*; manager II works 50 % of the shifts and he places orders will when the stock dips below 20% of the delivery_quantity; manager III works 20% of the shifts and he places orders when the stock is just under 5% of the delivery_quantity. Modify your code to account for this. What is the new optimal deliver_quantity? (Note: there is now no delivery_interval.) Hand in your modified code for grading.

PROBLEM 7.6. In the simulation code provided the days without tires are all charged at the same flat rate. Modify the code to account for the fact that some tires may have been sold before the stock ran out. For example, if there were 700 tires at the beginning of the day and 800 were sold the penalty for not filling 100 orders should be smaller than if the whole day were spent without tires.

PROBLEM 7.7. Adapt the Blackjack computer program in the Appendices to exploit knowledge of the Dealer's showing card when deciding when to stop getting new cards. Run the simulation 1000 times (i.e., 2000 decks of cards) picking my_stay_value based on the value of the Dealer's face card and compare the results. Compare your winnings with the provided code that does not consider the Dealer's face card. You should win a higher percentage after the modification.

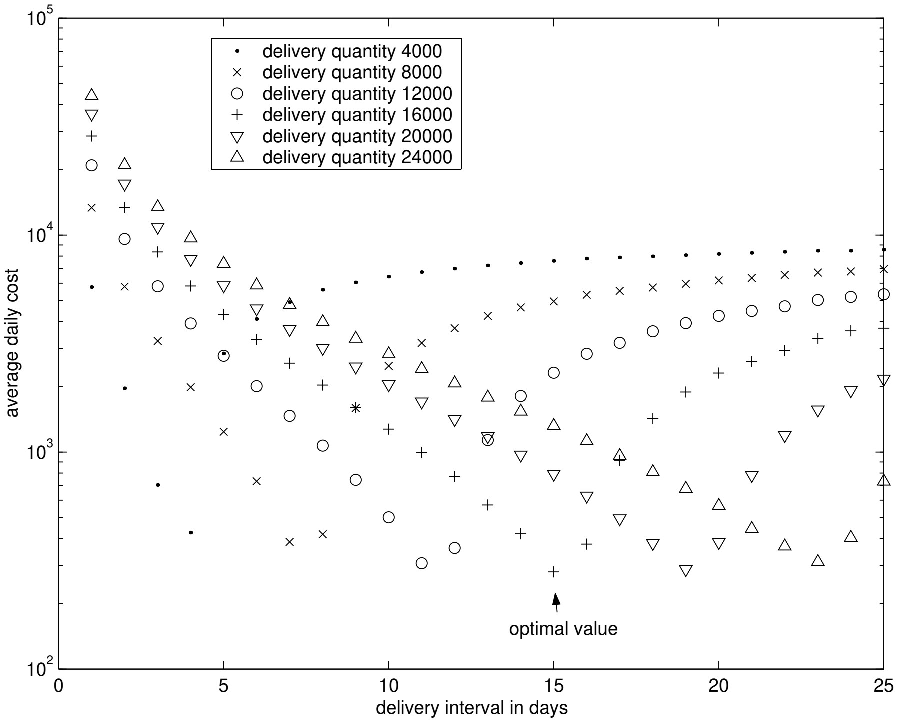

FIGURE 4. Average daily cost for a range of delivery intervals and delivery quantities. The minumum costs are themselves averaged over 365 day cycles and in turn these cycles are rerun 100 times and the result of the minimum daily cost averaged. The minimum average cost in these curves is (rounded to the nearest dollar) \$415, \$416, \$354, \$331, \$340 and \$341. A deliver interval of 15 days and quanity 16,000 tires appears optimal from our simulations.

PROBLEM 7.8. Experiment with the player's strategy for deciding whether to select an ace as a one or eleven based on the Dealer's showing card. Can you improve over your results in the previous problem?