Remark 2.4

In the case of continuous motions, from the material derivative of an integral formula (1.2.5) and the continuity equation (2.7) we get

\[
\frac{d}{dt} \int_D (pF) \, dV = \int_D \left[ \frac{d}{dt} (pF) + pF \Delta \vec{v} \right] \, dV = \int_D \left[ p \frac{dF}{dt} + F \left( \frac{d\rho}{dt} + p \Delta \vec{v} \right) \right] \, dV = \int_D p \frac{dF}{dt} \, dV.
\]

We will use this formula when deriving the balance equations.

Definition: Mass flux

Let's calculate the quantity of material through a surface \( S \) in an interval of time equal to \( t \) (see Figure 2.1). Let \( \Delta a \) be a surface element; the quantity of mass through \( \Delta a \) in \( \Delta t = 1 \) is contained in the cylinder with generator \( \vec{n} \Delta a \), as \( 1 \Delta a \) is the distance per unit time. With a sign convention that the flux is positive when \( \vec{v} \) and \( \vec{n} \) point in the same direction, or negative otherwise, we obtain for the flux of mass the formula

\[
\Phi = \int_S p \vec{v} \cdot \vec{n} \, da
\]

as \( dV = p \, dV \), and \( \Delta V = |\vec{v} \cdot \vec{n}| \Delta a \).

In fact, in this way we can calculate the flux of any variable having a density \( \Phi \); we get

\[
\Phi = \int_S \vec{v} \cdot \vec{n} \, da
\]
2.3. The Principle of the Variation of Momentum.

The momentum of a continuous system \( P \) with support \( D \) is

\[
\mathbf{H} = \int_\mathcal{P} \mathbf{v} \, dm = \int_\mathcal{D} \mathbf{p} \mathbf{v} \, dV. \tag{2.13}
\]

Postulate: a material system \( P \) moves such that at any time \( t \), the derivative of the momentum with respect to time is equal to the resultant of the external forces acting on \( P \),

\[
\frac{d\mathbf{H}}{dt} = \mathbf{R} \tag{2.14}.
\]

For a continuous medium we need to consider both contact forces of resultant \( \mathbf{R}^c \),

\[
\mathbf{R}^c = \int_\partial \mathbf{r} \, d\mathbf{a} \tag{2.15}
\]

and momentum

\[
\mathcal{M}^c = \int_\partial \mathbf{r} \times \mathbf{r} \, d\mathbf{a} \tag{2.16}
\]

and distance forces like gravitational attraction, of density \( \mathbf{f} \); their resultant force and resultant momentum are

\[
\mathbf{R}^d = \int_\mathcal{P} \mathbf{r} \, dm = \int_\mathcal{D} \mathbf{p} \mathbf{r} \, dV, \tag{2.17}
\]

and

\[
\mathcal{M}^d = \int_\mathcal{P} \mathbf{r} \times \mathbf{r} \, dm = \int_\mathcal{D} \mathbf{p} \mathbf{r} \times \mathbf{r} \, dV, \tag{2.18}
\]

respectively.

From (2.15) - (2.18) and (2.14) we get that

\[
\frac{d}{dt} \int_\mathcal{D} \mathbf{p} \mathbf{v} \, dV = \int_\partial \mathbf{r} \, d\mathbf{a} + \int_\mathcal{D} \mathbf{p} \mathbf{f} \, dV, \tag{4) DCD}, \tag{2.19}
\]
The above formula is valid for both continuous and discontinuous motions. For continuous motions only, if we take into account (2.10), we get the momentum equation under the form

$$\int_D \mathbf{F} \cdot d\mathbf{V} = \int_D \mathbf{F} \cdot d\mathbf{a} + \int_D \rho \mathbf{v} \cdot d\mathbf{V}$$  \hspace{1cm} (2.20)

2.4. The Principle of Variation of the Kinetic Momentum

The kinetic momentum of any part $P$ of a continuous material system $M$ can be defined as

$$\mathbf{K} = \int_M \mathbf{r} \times \mathbf{v} \cdot d\mathbf{m} = \int_D \rho \mathbf{r} \times \mathbf{v} \cdot d\mathbf{V}$$  \hspace{1cm} (2.21)

The principle of variation of the kinetic momentum: A material system $M$ moves such that, at any time $t$ and for any subdomain $P$ of $M$, the time derivative of the kinetic momentum equals the resultant momentum of external forces acting on $P$.

Analytically, we can write the above principle as

$$\frac{d}{dt} \int_D \rho \mathbf{r} \times \mathbf{v} \cdot d\mathbf{V} = \int_D \mathbf{F} \cdot d\mathbf{a} + \int_D \rho \mathbf{r} \times \mathbf{v} \cdot d\mathbf{V}$$  \hspace{1cm} (2.22)

and for continuous motions, using the formula (2.10), we get that

$$\int_D \mathbf{r} \times \mathbf{a} \cdot d\mathbf{V} = \int_D \mathbf{F} \cdot d\mathbf{a} + \int_D \rho \mathbf{r} \times \mathbf{v} \cdot d\mathbf{V}$$  \hspace{1cm} (2.23)

Cauchy's Lemma *)

If \( \bar{\tau} \) is continuous w.r.t. \( x^3 \), then at any \( x^3 \in \mathbb{D} \) we have
\[
\bar{\tau}(x^3, \bar{n}) = -\bar{\tau}(x^3, -\bar{n})
\] (2.24)

Proof

Consider a cylinder \( D \), centered at \( x^3 \), such that if reduces to the disc \( \Sigma \) when its height \( h \to 0 \) (see Figure 2.2). From (2.20) we have
\[
\int p \bar{n}^2 \, dv = \int \bar{\tau} \, da + \int \bar{f} \bar{n} \, d\nu
\]
and the volume integrals \( \int \) as well as the integrals on the surface of \( D \), the external surface approach zero as \( h \to 0 \). We are left with
\[
\int [\bar{\tau}(x^3, \bar{n}) + \bar{\tau}(x^3, -\bar{n})] \, da = 0
\]
and from the Fundamental Lemma we get that \( \bar{\tau}(x^3, \bar{n}) + \bar{\tau}(x^3, -\bar{n}) = 0 \) q.e.d.

Cauchy's Theorem

If \( \bar{\tau} \) is a continuous function on \( \mathbb{D} \) then there exists a tensor \( \overline{\tau}(x^3) \), defined on \( \mathbb{D} \), such that at any point \( p(x^3) \) we have
\[
\bar{\tau}(x^3, \bar{n}) = \overline{\tau}(x^3) \bar{n}^3
\] (2.25)
on components,
\[
\bar{\tau}_{ij}(x^3, \bar{n}) = \overline{\tau}_{ij} \bar{n}^i \bar{n}^j, \quad i, j = 1, 2, 3
\] (2.26).

Remark 2.4. Cauchy's theorem shows that \( \overline{\tau} \)
is a linear function of \( \bar{n}^3 \).

*) We consider the current (Eulerian) configuration, at an instant time \( t \), so we omit \( t \) from \( \overline{\tau}(x^3, \bar{n}, t) \).
Remark 2.5. \( \mathbf{T}(\mathbf{x}^2) \) characterizes completely the stress state at \( P(\mathbf{x}^2) \). Indeed, if we know \( \mathbf{T}(\mathbf{x}^2) \) we can then calculate the stress vector \( \mathbf{T} \) for any oriented surface of normal \( \mathbf{n} \).

Proof

We consider a tetrahedron centered at \( P \) and having three sides perpendicular to the coordinate axes (see Fig. 2.3). Denote by \( h \) the length of the perpendicular from \( P \) to the opposite side \( PP_2P_3 \).

We observe that the volume of the tetrahedron approaches zero when \( h \to 0 \). This is not a particular choice since the dependence of \( \mathbf{T} \) on \( \mathbf{n} \) is unique. Denote by \( \mathbf{n} \) the normal to \( PP_2P_3 \), and by \( \Delta a_i \), \( i = 1, 2, 3 \), the areas of the sides \( PP_2P_3 \), \( PP_1P_3 \), \( PP_1P_2 \). We have denoted by \( \mathbf{T}_j(\mathbf{x}^2) = \mathbf{t}(\mathbf{x}, \mathbf{v}_j) \). According to Cauchy's Lemma, we denote by \( -\mathbf{T}_j \) the stress vector in the side \( PP_2P_3 \); analogously, we can define \( -\mathbf{T}_2 \) and \( -\mathbf{T}_3 \). If \( \Delta a \) is the area of \( PP_2P_3 \), we have

\[
\Delta a_i = \mathbf{n} \cdot \Delta a, \quad i = 1, 2, 3 \tag{2.27}
\]

The volume of the tetrahedron is \( \frac{1}{6} h \Delta a \). We apply the principle of the variation of momentum with \( D \) the tetrahedron and we use the formula

\[
\int_D \mathbf{F}(\mathbf{x}^2) dV = (\mathbf{F}(\mathbf{x}) + \mathbf{E}) \int_D dV,
\]

where \( \lim \varepsilon = 0 \). We get, from (2.27),

\[
p(\mathbf{x}) \left[ \mathbf{a}_i(\mathbf{x}) - f(\mathbf{x}^2) - \varepsilon \right] \quad \frac{1}{6} h \Delta a = \left[ \mathbf{t}(\mathbf{x}, \mathbf{n}) + \mathbf{E}_n \right] \Delta a - \sum_{j=1}^{3} \left[ \mathbf{T}_j(\mathbf{x}) + \mathbf{E}_j \right] \Delta a_j.
\]
with \[ \lim_{n \to 0} (\xi, \eta, \zeta) = 0. \]

Using (2.27), dividing by $\Delta a$ and taking the limit $n \to 0$, we get
\[
\overrightarrow{t}(\mathbf{x}, \mathbf{n}) = T_{ij}(\mathbf{x}) n_j. 
\]
(2.28)

After projection of (2.28) on $0x_i$ we get (2.26).

**Remark 2.6.**
The formula (2.26) is the proof that $T$ is a tensor. It is the stress tensor, the first tensor to be defined in sciences. The name "tensor" comes from "tension".

**Remark 2.7.**
Let's project $t(\mathbf{x}, \mathbf{n})$ along a direction of unit vector $\mathbf{m}$. We have that
\[
\mathbf{m} \cdot \overrightarrow{t} = \overrightarrow{m} \cdot T \mathbf{n} = T_{ij} m_i n_j. 
\]
(2.29)

This formula helps us to show that if we change the basis from $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ to another orthogonal basis of $\mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_3'$, the quantity $\overrightarrow{t}$ changes according to the formula
\[
\overrightarrow{t} = \mathcal{Q} \overrightarrow{t} \mathcal{Q}^T. 
\]
(2.30)
where $\mathcal{Q}$ is the change of base matrix. Equivalently,
\[
T_{ij}' = \mathcal{Q}_{ke} \mathcal{Q}_{ij} T_{ij}. 
\]
(2.30')

If we take $\mathbf{m} = \mathbf{n}$ we get the formula for the normal stress
\[
\mathcal{N} = \mathbf{n} \cdot \overrightarrow{t} = \mathbf{n} \cdot T \mathbf{n} = T_{ij} n_i n_j. 
\]
(2.31)

**Question.** Find those directions $\mathbf{n}$ for which $\mathcal{N}$ takes extremal values.