Problem 1: Show that the mappings described below are linear:
(a) \( T : \mathbb{C} \rightarrow \mathbb{C} \) (with \( \mathbb{C} \) regarded as a vector space over \( \mathbb{R} \)) mapping a complex number into its conjugate
(b) \( T : P_3 \rightarrow P_3 \) defined as \((Tp)(t) = p(t+1) - p(t) + \int_{-1}^{1} s^2 p(s) \, ds\)

Problem 2: Investigate the validity of the following statement and prove it if it is true, give a counterexample if it is false: If \( I \) is a non-zero scalar linear function on a (not necessarily finite-dimensional) linear space \( X \), and if \( \alpha \) is an arbitrary scalar, does there necessarily exist a vector \( x \in X \) such that \( I(x) = \alpha \)?

Problem 3: Show that if \( \dim X = 1 \) and \( T \in \mathcal{S}(X, X) \) then there is \( k \in \mathbb{K} \) such that \( Tx = kx \) for all \( x \in X \).

Problem 4: Suppose that \( U \) and \( V \) are finite-dimensional linear spaces and \( S \in \mathcal{S}(V, W) \), \( T \in \mathcal{S}(U, V) \). Show that \( \dim N_{ST} \leq \dim N_S + \dim N_T \).

Problem 5: Let \( T : \mathbb{C}^3 \rightarrow \mathbb{C}^3 \) be defined as
\[ T((a_1, a_2, a_3)) = (a_1 - a_2 + ia_3, 2a_1 + ia_2, (2 + i)a_1 - a_3) \]
(a) Verify that \( T \) is a linear map
(b) Find \( R_T \) and \( N_T \) (by giving bases for both).

Problem 6: Show that if \( X \) is a finite-dimensional space then the space \( L(X, X) \) of all linear maps of \( X \) into \( X \) is finite-dimensional. Find the dimension of \( L(X, X) \).

Problem 7: Let \( T : P_n \rightarrow P_n \) be the linear map such that \( Tp(t) = p(t + 1) \). Show that if \( D \) is the differentiation operator then
\[ T = 1 + \frac{D}{1!} + \frac{D^2}{2!} + \ldots + \frac{D^{n-1}}{(n-1)!} \]

Problem 8: If \( A \) is a linear map on an \( n \)-dimensional linear space, then there exists a non-zero polynomial \( p \) of degree \( \leq n^2 \) such that \( p(A) = 0 \).

Problem 9: Let \( \theta \) be a real number. Show that the following two matrices are similar over the field of complex numbers:
\[
\begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}, \quad \begin{bmatrix}
e^{i\theta} & 0 \\
0 & e^{-i\theta}
\end{bmatrix}
\]

Problem 10: Let \( T \) be a linear operator on \( \mathbb{R}^2 \) defined by \( T(a_1, a_2) = (-a_2, a_1) \). Prove that for every real number \( c \) the operator \( (T - cI) \) is invertible (without the use of determinants or eigenvalues).