(IVP): $c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & 2 \\ 0 & 4 & 4 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -2 & 2 & 1 \\ 0 & 4 & 4 & -1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -2 & 2 & 1 \\ 0 & 0 & 8 & 1 \end{array}\right) \Rightarrow \begin{cases} c_1 = 10/8 = 1.25 \\ c_2 = -3/8 = -0.375 \\ c_3 = 1/8 = 0.125 \end{cases}$$

$\Rightarrow \boxed{y(t) = c_1 + c_2 e^{-2t} + c_3 e^{2t}}$

$\Rightarrow \boxed{y(t) = \frac{1}{8} - \frac{3}{8} e^{-2t} + \frac{10}{8} e^{2t}}$

Other method: use characteristic equation:

$\lambda^3 - 4\lambda = 0 \Rightarrow \lambda(\lambda^2 - 4) = 0 \Rightarrow \lambda_1 = 0,\ \lambda_2 = -2,\ \lambda_3 = 2 \Rightarrow$

$\Rightarrow y(t) = c_1 e^{0t} + c_2 e^{-2t} + c_3 e^{+2t}$ and so on

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5) $\mathcal{L}(y'' - y) = \mathcal{L}[e^{2t}] \Rightarrow \mathcal{L}[y''] - \mathcal{L}[y] = \mathcal{L}[e^{2t}] \Rightarrow$

(a) $\Rightarrow s^2 \mathcal{L}[Y] - s y(0) - y'(0) - \mathcal{L}[Y] = \frac{1}{s-2} \Rightarrow$ with $\mathcal{L}[Y] = Y(s)$, $y(0) = 0$, $y'(0) = 1$ we have

$s^2 Y(s) - 1 - Y(s) = \frac{1}{s-2} \Rightarrow (s^2 - 1) Y(s) = \frac{1}{s-2} + 1$

$\Rightarrow (s^2 - 1) Y(s) = \frac{s+1}{s-2} \Rightarrow Y(s) = \frac{s+1}{(s-2)(s^2-1)} = \frac{s+1}{(s-2)(s-1)(s+1)} =$

$\Rightarrow Y(s) = \frac{1}{(s-2)(s+1)}$ 15p

(b) You can use the results from (a) or standard method.
With Laplace transform

$Y(s) = \frac{1}{(s-2)(s+1)} = \frac{1}{3}\left[\frac{1}{s-2} + \frac{1}{s+1}\right] \Rightarrow$

$\Rightarrow y(t) = \mathcal{L}^{-1}\left[\frac{1}{3}\left(\frac{1}{s-2} + \frac{1}{s+1}\right)\right] = \boxed{\frac{1}{3}\left(e^{+2t} - e^{-t}\right)}$ 15p

Or: $y_G = y_p + y_H$, where $\begin{cases} y_H(t) \text{ is the solution of } y'' - y = 0 \\ y_p(t) \text{ is a particular solution.} \end{cases}$

$y'' - y = 0 \Rightarrow \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1, \Rightarrow \boxed{y_H(t) = c_1 e^{t} + c_2 e^{-t}}$

$y_p(t) = A e^{2t} \Rightarrow A = \frac{1}{3}$

$\Rightarrow \begin{cases} y_G(t) = c_1 e^{t} + c_2 e^{-t} + \frac{1}{3} e^{2t} \\ y(0) = 0,\ y'(0) = 1 \end{cases} \Rightarrow \begin{cases} c_1 = 0 \\ c_2 = -\frac{1}{3} \end{cases}$

$\Rightarrow \boxed{y(t) = \frac{1}{3}\left(e^{2t} - e^{-t}\right)}$