Solutions, Final exam

M340 Fall 2005

(1) $M(x,y) = 2xy^2 + 2y$; $\frac{\partial M}{\partial y} = 2xy + 2 = \frac{\partial N}{\partial x}$ $\boxed{\text{exact}}$ (5pt).

$N(x,y) = 2x^2y + 2x$

Solution: $\begin{cases} \frac{\partial \psi}{\partial x} = 2xy^2 + 2y \Rightarrow \psi(x,y) = x^2y^2 + 2xy + C(y) \Rightarrow \\ \frac{\partial \psi}{\partial y} = 2x^2y + 2x \quad \Rightarrow \frac{\partial \psi}{\partial y} = 2x^2y + 2x + c'(y) = 2x^2y + 2x \Rightarrow \end{cases}$

$\Rightarrow c'(y) = 0 \Rightarrow C(y) = c \Rightarrow \psi(x,y) = x^2y^2 + 2xy$

$\Rightarrow \boxed{x^2y^2 + 2xy = c}$ (15p) is the implicit form of the solution.

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(2) a) $\begin{cases} y' + \frac{4}{t}y = \frac{1}{t^3}e^{-t} \\ y(-1) = 0 \end{cases} \Rightarrow \boxed{(-\infty, 0)}$ (10p)

b) $\boxed{\mu(t) = t^4}$ (10p) $\Rightarrow \boxed{y(t) = \frac{(-t-1)e^{-t} + c}{t^4}}$ (5p),

$y(-1) = 0 \Rightarrow \boxed{c = 0} \Rightarrow \boxed{y_p(t) = (-t-1)e^{-t}/t^4}$ (5p)

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(3) (a) $y'' - 2y' - 3y = e^{-t}$ | $y'' - 2y' - 3y = 0 \Rightarrow \lambda^2 - 2\lambda - 3 = 0$, $\lambda_1 = -1, \lambda_1 = 3$

$\Rightarrow \{e^{-t}, e^{3t}\}$ FSS $\Rightarrow y_p = Ate^{-t}$

$\Rightarrow \boxed{y_p(t) = -\frac{1}{4}te^{-t}}$ (10p) $\Rightarrow y' = A(e^{-t} - te^{-t})$; $y'' = A(-2e^{-t} + te^{-t}) \Rightarrow A = -\frac{1}{4}$

(b) $y'' + 4y = 3\sin t$ $y'' + 4y = 0 \Rightarrow \lambda^2 + 4 = 0 \Rightarrow \lambda = \pm 2i \Rightarrow$

$\Rightarrow \{\cos 2t, \sin 2t\}$ FSS

$\Rightarrow y_p(t) = A\cos t + B\sin t \Rightarrow 3A\cos t + 3B\sin t = 3\sin t \Rightarrow$

$\Rightarrow A = 0, B = 1$

$\Rightarrow \boxed{y_p = \sin t}$ (10p)

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(4) $y''' - 4y' = 0 \iff \begin{cases} y' = u \\ u' = v \\ v' = 4u \end{cases} \iff \frac{d}{dt}\begin{pmatrix} y \\ u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 4 & 0 \end{pmatrix}\begin{pmatrix} y \\ u \\ v \end{pmatrix}$.

char. polynomial: $\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 0 & 4 & -\lambda \end{vmatrix} = (-\lambda)\begin{vmatrix} -\lambda & 1 \\ 4 & -\lambda \end{vmatrix} = (-\lambda)(\lambda^2 - 4)$

$\lambda_1 = 0, \lambda_2 = -2, \lambda_3 = 2$

Eigenvectors: $v^{(1)} = [1, 0, 0]$, $v^{(2)} = [1, -2, 4]$, $v^{(3)} = [1, 2, 4]$

$\Rightarrow$ general solution $\begin{pmatrix} y(t) \\ u(t) \\ v(t) \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2e^{-2t}\begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} + c_3e^{2t}\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$