Computing with Finite Matrix Groups

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State of the Art

There are practical methods (GAP and Magma) for computing

Task	Permutation	Matrix
Order, Membership	Stabilizer Chain	Matrix Group Recognition
Composition Structure		
Homorphisms		
Centralizers, Normalizers	Backtrack	Want !
Conjugating Elements		
Classes of Elements		Extend Solvable Radical Method
Subgroups	Solvable Radical / Trivial Fitting	
Isomorphism Test		

PcGroups

Step1: Matrix Group Recognition

Matrix Group Recognition finds actions and thus obtains a composition tree

$$
\begin{tikzcd}[column sep=small]
 & & G \quad \text{with } \varphi \arrow[dl] \arrow[dr] & & \\
 & N = \ker\varphi \leq G \text{ with } \psi \arrow[dl] \arrow[dr] & & F = \mathrm{Image}(\varphi) \text{ with } \chi \arrow[dl] \arrow[dr] & \\
\ker\psi \leq N & & \mathrm{Image}(\psi) & \ker\chi \leq F & & \mathrm{Image}(\chi)
\end{tikzcd}
$$

At each node, we can evaluate the homomorphism (by acting on the objects of the underlying decomposition) and have generators for the kernel.

Leafs

Each leaf of the tree is a simple group.

We know its type and have an isomorphism to a natural representation. (Sssume we know *everything* about the simple groups.)

The tree thus represents a composition series of *G*.

We know the subgroups in this series and for each subgroup the homomorphism on its simple quotient.

Step 2: Radical and its Quotient

To use the solvable radical method, we need to find $R = \mathrm{Rad}(G) \triangleleft G$, the largest solvable normal subgroup, and an effective homomorphism $\varrho: G \to G/R$.

$\mathrm{Soc}(G/R)$ is direct product of simple nonabelian groups and (up to isomorphism) $G/R \leqq \mathrm{Aut}(\mathrm{Soc}(G/R))$.

So ϱ should be the action of G on this socle. But the socle factors are spread over the composition series.

Reconstructing the Socle Action

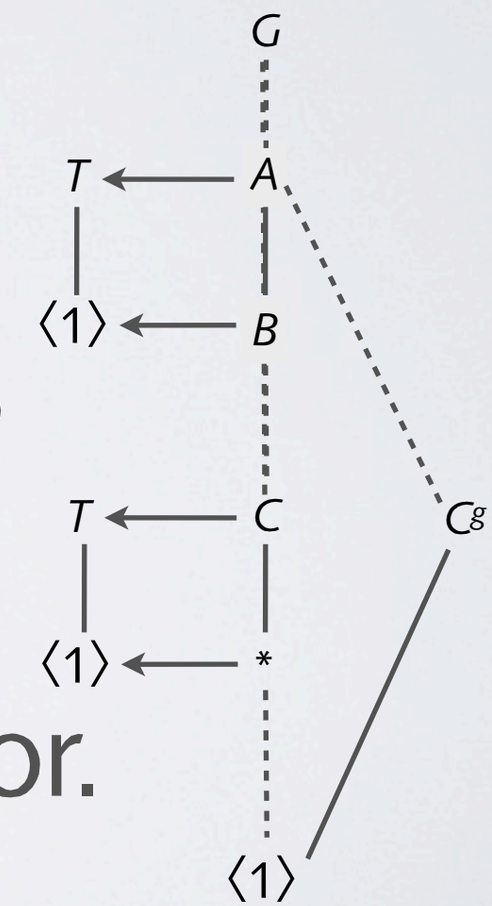
Let C be a subgroup in the composition series, $C \to T$ simple nonabelian quotient in series. If C is deepest in series, elements of C represent a **single** factor of this socle.

Conjugation by $g \in G$ will map C to C^g.

In chain, C^g maps to quotient A/B of same isomorphism type.

A/B represents another socle factor.

We thus can act on $\mathrm{Soc}(G/R)$.



Combining Actions

The G on non-abelian composition factors of one type T yields a homomorphism $\alpha: G \to (\mathrm{Aut}\ T) \wr S_n$. Image is permutation group (or matrix group). Combine to $\varrho = \alpha_1 \times \ldots \times \alpha_m$ into direct product.

This is the action of G on $\mathrm{Soc}(G/R)$. Thus $\ker \varrho = R$.

If the image is a permutation group, use existing methods for computation.

Layering the Radical

Conjecture: Solvable matrix group R usually has a short orbit on vectors or submodules.

If no large primes: $\max(12,n)\cdot(q^{(n/2)}+1)$

Submodules for R', R'',... give candidates.

Algorithm by Sims (solvable BSGS) finds series $G \rhd R = R_0 \rhd R_1 \rhd \cdots \rhd \langle 1 \rangle$ with R_i/R_{i+1} elementary abelian, coefficients in these vector spaces (PCGS).

Step 3: Working with Subgroups

To avoid evaluating ϱ represent $U \leqq G$ by:

- An induced PCGS (think: REF for matrix) for $U \cap R$.
- Generators $u_i \in U$ s.t. $U = \langle U \cap R, u_1, u_2, \ldots \rangle$
- Images u_i^{ϱ} as elements of $G/R \leqq D$.

Element test in U then first tests in U^{ϱ}. Then divide off and test in $U \cap R$.

Analogously, for any $x \in G$ in the algorithm also maintain its image $x^{\varrho} \in G/R$.

Step 4: Lifting

We can now proceed essentially in the same way as for permutation groups:

Assume we know the result in $G/R = G/R_0$. (E.g. by permutation group methods, if this is a permutation group.)

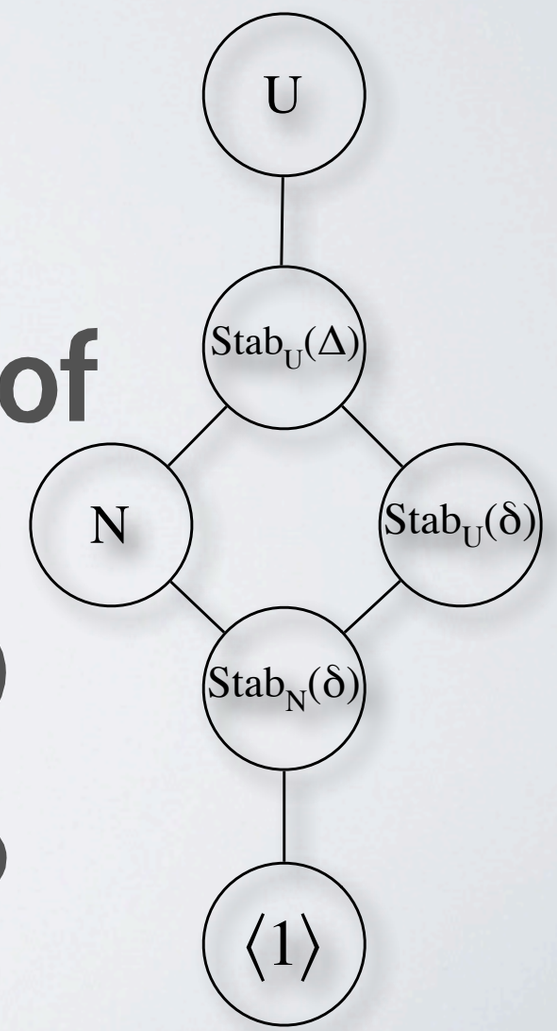
Now go repeatedly from G/R_i to G/R_{i+1} until we reach $R_k = \langle 1 \rangle$.

Each step reduces to orbit calculations for an (affine) action on R_i/R_{i+1}.

Orbit/Stabilizer Algorithm

When calculating orbit/stabilizer of δ under U (this will be a basic operation)

- Calculate the orbit Δ of δ under $N = U \cap R \triangleleft U$ and the stabilizer $V \leqq U \cap R$ of δ. Δ is a U-block.
- Calculate the orbit and stabilizer **of Δ** under U by computing in $U^{\underline{e}}$. (Represent Δ^u by single element.)
- Correct generators of $\mathrm{Stab}_U(\Delta)$ to get $\mathrm{Stab}_U(\delta)$ as complement.



Step 5: Implementations

New interface for solvable radical code in GAP 4.7. Used by new ConjugacyClasses/Centralizer/Canonical Conjugate routine. (MECKY/NEUBUESER, SOUVIGNIER/HOLT/CANNON, H.)

(Backtrack centralizer is often faster, but canonical element is nice.)

Experimental implementation of this interface for matrix groups, using `recog` package (NEUNHOEFFER, SERESS).

Applicable to matrix groups of considerable size.

Conjugacy Classes Runtimes

Times in Seconds on a 2.6GHz MacPro.

Group	Order	deg	q	#Classes	t_{Setup}	t_{Calc}
$(GL_2(5) \wr S_3) \perp_6 (L_2(11) \wr S_3)$	190768545792000000	21	5	1235200	17	22886
$(GL_2(5) \wr S_3) \perp_2 (L_2(11) \wr S_3)$	572305637376000000	21	5	503808	22	9078
$2^{9+16}.S_8(2)$	1589728887019929600	394	2	703	455	998
$3^{1+12}.2Suz.2$	2859230155080499200	78	3	253	427	76
$5^9:(GL_3(5) \times GL_3(5))$	4324500000000000000	6	5	18464	3	361
$(6.A_5) \wr S_5$	56422198149120000000	30	25	526473	27	2863
$3^{15}:(M_{11} \wr S_3)$	42770626907728896000	16	3	3200	14	129
$2^{1+22}.Co_2$	354883595661213696000	1045	2	448	654	4790
$((2^2 \times 3).U_6(2)) \wr S_2$	2435952824419268689920 0	54	4	77814	41	1281
$11^9:(SL_3(11) \times SL_3(11))$	5364074707031250000000000	6	11	20759	5	2819
$(5^{3\bullet} L_3(5)) \wr S_{3.}$	6032677500000000000000000	75	5	12200	97	1270
$2^{44}:(M_{11} \times (2^4:(S_3 \times S_3)))$	2480816141360352083312640	15	2	10759	5	4167
$(3^{10}:(M_{11} \wr 2)) \perp_2 (J_2 \wr 2)$	2709670423891673088000000	83	3	127764	44	13729
$7^{12}:(SL_3(7) \times SP_4(7))$	2155671513942745138421760 0	7	7	7701	12	2670

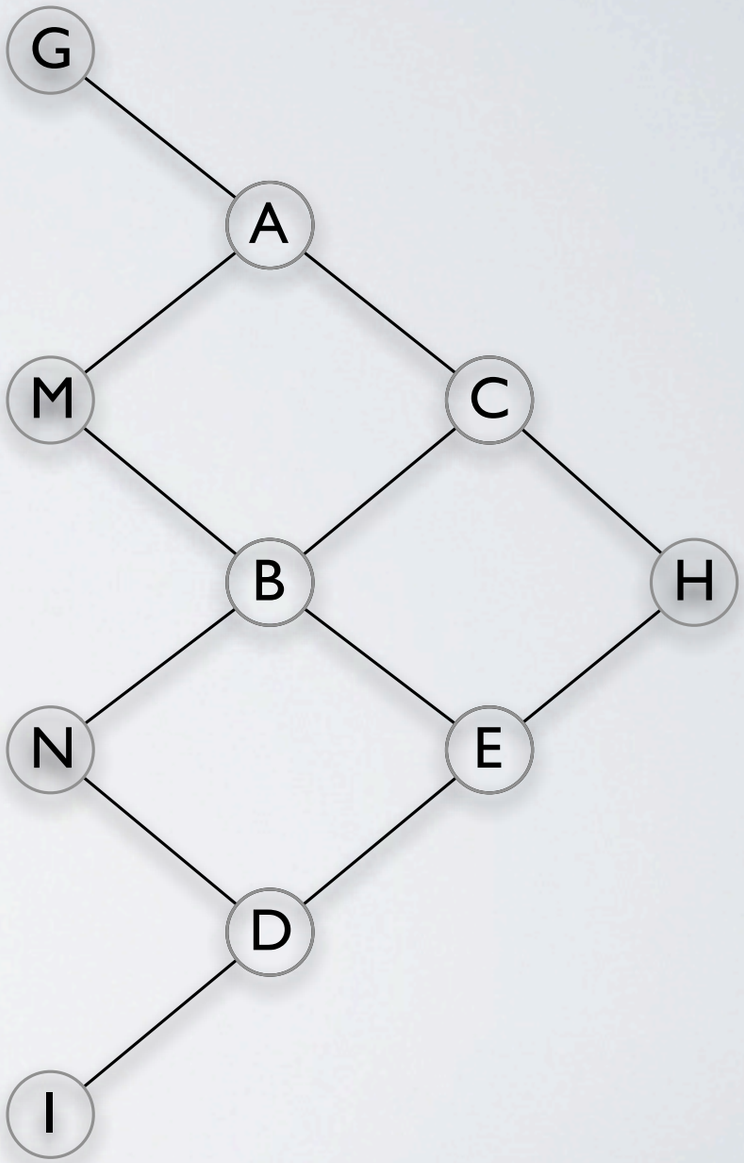
Subgroup Normalizers

Experimental implementation of Subgroup Normalizer (following GLASBY/SLATTERY method for PcGroups.)

To normalize H in G, when $M,N \triangleleft G$, normalize closures/intersections in order $A,B,C,\ldots$

Up step: Stabilize complement (act on cohomology).

Down Step: Stabilize Subspace (Scarily long orbits, centralizer and induced automorphisms.)



Constructive Existence Proof of Implementation

```
gap> g:=AtlasSubgroup("J4",IsMatrixGroup,1);  #2^11.M24
<matrix group of size 501397585920 with 2 generators>
gap> ff:=FittingFreeLiftSetup(g);;          # very fast
gap> u:=Subgroup(g,[g.1,g.1^g.2]);;
gap> Size(u);
22
gap> NormalizerViaRadical(g,u);
<matrix group of size 220 with 4 generators>

gap> u:=HallViaRadical(g,[3]);       # Joint work with Eick
[ <matrix group of size 27 with 3 generators> ]
gap> n:=NormalizerViaRadical(g,u[1]);time;
<matrix group of size 432 with 7 generators>
6583
```

Somewhat larger example

```
gap> g;  # Max in BM,  dimension 813 over GF(2), 4ms/prod.
2^(2+10+20).(M22:2 x S3)
gap> Size(g);
22858846741463040
gap> ff:=FittingFreeLiftSetup(g);; # ~10 minutes
#I  Used Base Points[ ...] Lengths [ 3, 2, 32768, 32, 4096 ]
gap> ff.radical;
<matrix group of size 25769803776 with 34 generators>
gap> ff.pcisom;
Pcgs([ <an immutable 813x813 matrix over GF2>, [...]
  <an immutable 813x813 matrix over GF2> ]) -> Pcgs(
[ f1, f2, f3, f4, f5, f6, f7, f8, f9, f10, f11, f12, f13, f14, f15,
  f16, f17, f18, f19, f20, f21, f22, f23, f24, f25, f26, f27, f28,
  f29, f30, f31, f32, f33, f34 ])
gap> u:=HallViaRadical(g,[3]);time;
[ <matrix group of size 27 with 3 generators> ]
36697
```

...continued

```
gap> n:=NormalizerViaRadical(g,u[1]);
#I  Radsize= 9 index 1
#I  abelian factor 2: 25769803776->12884901888 central:true
#I  down
#I  module of dimension 1 subspace 0
#I  up 0: on 2 cobounds:1
#I  abelian factor 3: 12884901888->4294967296 central:false
#I  down
#I  module of dimension 1 subspace 1
#I  abelian factor 4: 4294967296->4096 central:false
#I  down
#I  module of dimension 2 subspace 0
#I  module of dimension 18 subspace 0
#I  module of dimension 20 subspace 0
#I  up 2:3 on 1048576 cobounds:1048576
#I  up 0: on 1048576 cobounds:1
```

...continued more

```
#I  abelian factor 5: 4096->4 central:false
#I  down
#I  module of dimension 1 subspace 0
#I  module of dimension 2 subspace 0
#I  module of dimension 10 subspace 0
#I  up 2:3 on 1024 cobounds:1
#I  up 0: on 1024 cobounds:1
#I  abelian factor 6: 4->1 central:false
#I  down
#I  module of dimension 2 subspace 0
#I  up 2:3 on 4 cobounds:4
#I  up 0: on 4 cobounds:1
<matrix group of size 3456 with 9 generators>
gap> time;   # about 50 minutes
2828467
gap> Size(g)/Size(last);
6614249635840
```