Notes on Computational Group Theory

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The heart of such functionality is the constructive recognition. Methods for this have been proposed for many classes of simple groups. Some of these assume that G is alreeady given in a particular representation (such as: S_n in the action on pairs of numbers), some do not assume anything about G, but only the fact that we can do element arithmetic and compare elements and have a bound for $|G|$ (a so-called "black-box" group).

The crucial step of such an algorithm then is to recreate the underlying geometry or combinatorics of the group (in the case of (P)GL the vector space) based on properties of group elements.

For example $\mathrm{GL}_n(q)$ is generated by matrices A such that $A - A^0$ has exactly one nonzero entry. Such elements are called "transvections" in the geometric context. We now want to

- Find such elements (and prove that we can find them using random search in reasonable time)
- Identify such elements using only black-box operations.

Doing this requires substantial knowledge about classical groups.

C_2 M is the direct sum of isomorphic subspaces $M = \bigoplus_{V \in \mathcal{V}} V$ and the action of G permutes these subspaces.
C_4 M is the tensor product of two nonisomorphic submodules.
C_7 M is the tensor product of isomorphic spaces $M = \bigotimes_{V \in \mathcal{V}} V$ which are permuted under the action of G.
C_3 Up to scalars, we can write G in smaller dimension over a larger field.
C_5 Up to scalars we can do a base change to write G over a smaller field.
C_6 The group G is normalizing a p-group of form $p.p^k$.
C_8 The group G is stabilizing a bilinear or quadratic form.

Note that most of the classes yield an obvious factor group (and suitable action for obtaining an epimorphism onto this factor).

Algorithmically, algorithms exist that with high probability will recognize the various cases (for example the MeatAxe recognizes class C_1). If no class is recognized we assume that G is almost simple and then try to process it as an almost simple group.

VI.3. Constructive Recognition

Assume that we have a group G given by generators of which we believe that it is almost simple. For this group we would like to the following:

Recognition: Find out the isomorphism type (with high probability). We can do this de facto (using **heavy theory** by looking at the distribution of orders for (pseudo-)random elements.

Constructive Recognition: Given the (likely) isomorphism type of G construct an effective homomorphism (we have a method to compute images and pre-images of elements) from G to a "gold-plated" nice version (typically the "natural" definition) of this group.

Work in the nice version: Using the treasure of knowledge built up in theory, we want to find out "everything" (classes, subgroups, presentation, &c.) about the nice group and use the isomorphism to translate the information back to G.

Verify the isomorphism: Express known generators of the nice groups as words in images of the generators of G. Express the images of the generators of G as words in the nice generators. Evaluate a presentation for the nice group in G.

While we are describing these processes in the context of matrix groups the transfer of data buit using theory from a "nice" group back to an almost simple group G often is what is needed in lifting algorithms to deal with the radical factor group.

builds on a composition series and decomposition into generators for the simple factors.

As in this situation many of the algorithms we use will use (Monte Carlo) randomization and might return a wrong result. As with permutation groups we therefore use the composition series to determine a presentation and verify relators.

A second feature of this process is that we will be working in a homomorphic image, the homomorphism defined via some action, and will have to take pre-images of elements under the homomorphism. As we don't have an easy way of decomposing into generators we therefore keep track of all operations done in the homomorphic image. We therefore know for every element x of this iomage how to express x as word in the generators. To obtain the pre-image of x we then simply evaluate this word in the original groups generators.

PERFORMANCE 169: De facto we will not store words, but "straight line programs" which have smaller storage and faster evaluation. The ide is essentially to store an expression such as $b((ab)^4 * b)^2 * (ab)^2$ not as word $bababab^2ababab^2abab$ but as "expression tree", storing $c = ab$, $d = (ab)^2 = c^2$, $e = (ab)^4 = d^2$ $f = eb$ and then express the word as bf^2d.

As with permutation groups there has been much interest in complexity aspects of these algorithms: Eventually we want a low-degree polynomial complexity in n and $\log(q)$. One potential difficulty with this is the case of large order cyclic factors. To test membership in such a group we need to solve a discrete logarithm problem, which is a known hard problem.

VI.2. Aschbacher's theorem

In its original form Aschbacher's theorem [Asc84] is a description of maximal subgroups of a matrix group. Since the full matrix group is $\mathrm{GL}_n(q)$ we can — analogous to theorem 79 — also read it as a statement about matrix groups that are not the full general linear group.

In the following we assume that $G \leq \mathrm{GL}_n(q)$ is a matrix group and $M = \mathbb{F}_q^n$ its natural module.

The theorem defines a series of classes $C_1, \ldots, C_8$ of subgroups, each offering a reduction of the natural module. The (very technical) proof is essentially to show that if a subgroup is not in classes C_1 to C_8 it must be in the class (called C_9) of almost simple groups.

The classes are roughly[1] defined as follows:

C_1 M is reducible.

[1] I'm leaving out various technical conditions

Preface

This are lecture notes I prepared for a course on Computational Group Theory which I taught in Spring 2006 at Colorado State University. The audience consisted of graduate students in their second year and later. All had taken a one year algebra sequence the year before and a course on representation theory in the previous semester (which explains the lack of any description of representation theoretic methods in these notes).

My aim in this course was to give an overview over *most* of computational group theory from the point of view of understanding the principle behind calculations and understand what kinds of calculations are easy, hard or infeasible.

In many cases however the presentation, prominence given to particular algorithms or classes of groups, or depth of description is hopelessly biased by my personal preferences and the desire to use this course to prepare students for dissertation work with me.

In particular, as only few of the students had a background in computer science, I decided to essentially eliminate all references to complexity and in a few cases (which I hope all carry an explicit warning about this) even replace polynomial time algorithms with potentially exponential ones as long as the run time in practice is not too bad.

Another main divergence from "classical" descriptions is the lack of a chapter on polycyclic presentations. Instead these are treated with their arithmetic as a special case of rewriting systems, in their algorithms in the more general area of "lifting" algorithms using homomorphic images.

As all the students had taken a course on representation theory with me before (in which we studied the MeatAxe) these notes also lack any description of computational representation theory while referring to it in a few places.

I had initially decided to use Derek Holt's marvellous "Handbook of Computational Group Theory" [HEO05] as textbook. However I found that it is often quite detailled – a terrific fact if one wants to implement the algorithms, but sometimes necessitating more time for explanation than a one-semester course can allocate.

Besides Holt's book I have freely borrowed from and am indebted to Ákos Serres' work on permutation groups [Ser03], Charles Sims' tome on finitely presented groups [Sim94], lecture notes by Peter Neumann [Neu87] and notes I took in lectures of my advisor, Joachim Neubüser.

I apologize in advance if these references are not always explicitly listed, but after all these are lecture notes. Similarly I have not aimed to make the bibliography exhaustive. There are a few references to the research literature but I apologize for any ommissions.

You are welcome to use these notes freely for your own courses or students – I'd be indebted to hear if you found them useful.

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CHAPTER VI

Group Recognition and Matrix groups

We have seen already one way of working with matrix groups by considering them as permutation groups on a set of vectors. While thius approiach works it can yield an exceedingly large degree and thus is not feasible for larger examples.

Instead, when given a matrix group $G \leq \mathrm{GL}_n(q)$, we want to determine a composition series (or chief series) of G. With such a series in hand the methods of the previous chapter can become feasible.

(I should mention that much of this chapter is still subject of current research and therefore comparatively little is available in implementations.)

VI.1. Towards a Composition Series

The basic idea is to mimic the approach for permutation groups II.6:

> Given a matrix group G, prove that G is simple or find a homomorphism (which we can evaluate) $\varphi\colon G \to H$ such that H is a matrix group of degree not larger than G and that $N := \mathrm{Kern}\,\varphi > \langle 1 \rangle$.

For permutation groups the crucial step towards this was the reduction to primitive groups and the O'Nan-Scott theorem describing the structure of primitive groups. For matrix groups we will use reducibility of the natural module and Aschbacher's theorem (section VI.2).

There is the additional difficulty of how to obtain the kernel N of a homomorphism. For permutation groups we could do this using a stabilizer chain for which we don't have a feasible analogue.

Instead we will first process G/N to the point where we have determined a composition series and from this a presentation for G/N. We then evaluate the relators for G/N in preimages of the generators. This will yield normal subgroup generators for N.

We will then assume that N is generated by "a few" G-conjugates of these normal subgroup generators. To verify correctness we therefore finally need to show that the group generated by these conjugates is normal in G which we can do once we have an element test for N. Such a test again

CHAPTER I

Basics

I.1. What is Computational Group Theory

Computational Group Theory (CGT) is the study of algorithms for groups. It aims to produce algorithms to answer questions about concrete groups, given for example by generators or as symmetries of a certain algebraic or combinatorial structures.

Interest in this comes from (at least) three areas:

- Interest in developing algorithms: Can we actually calculate the objects we define theoretically in our algebra courses?
- Concrete questions about concrete groups: We are interested in a particular group and we want to find out more about it. Early examples of this happened in the classification of the finite simple groups, when theoreticians predicted the existence of certain groups and then a lot of effort was needed to construct these groups and determine their properties.

  Of course users here are not restricted to group theorists. For example a chemist might want to find out some properties of the symmetry group of a differential equation, in the same way as she would use Maple to solve an integral.
- Complexity theory (which is a somewhat surprising area to come up). **The** famous problem in theoretical computer science is the question whether P=NP, i.e. whether for any problem for which we can *verify* a solution quickly (quickly here means: "polynomial runtime") we also can *find* a solution quickly. (This is one of the Millennium problems for whose solution $\$10^6$ have been on offer.) Typical cases of this are "puzzle" type problems, such as the "Traveling Salesman" problem. One particular intriguing problem of this kind is "Graph Isomorphism", i.e. the question whether two graphs, given by their adjacency matrix, are in fact isomorphic. This problem seems to lie "between P and NP" and thus might be a good bellwether for determining the relation between these problem classes.

A graph isomorphism however is simply a permutation of the vertices, preserving the edge incidences. Thus there has been the hope that permutation group methods can help in studying this problem.

Indeed in 1982, G. Luks[Luk82] solved the problem in polynomial time for a particular class of graphs. His solution uses substantial (computational) group theory. Since then there has been much interest in CGT from theoretical computer science.

This course is intended as an introduction to computational group theory which will lead you to a level where you could starting to read the research literature. The chosen textbook by Holt [HEO05] covers this material and much more.

I.2. A short introduction to GAP

Some computational group theory methods are implemented in many computer algebra systems, but there are two systems which specialize on it: GAP and Magma. We will use GAP.

See the first homework sheet.

I.3. Memory

As often in computation we could buy runtime at the expense of memory. In fact for many larger calculations memory use is more of an obstacle than run time. (You can wait a day longer but this still won't increase your systems memory.)

To understand some of the choices or trade-offs we will be making, it is useful to understand a bit about memory use for different objects:

Numbers:: A computer stores numbers in base 2, so we need $2 \cdot \log_2(n)$ bits to represent a signed number of magnitude n. (In fact we typically allocate memory in chunks of 4 bytes on a 32 bit system.)

Small Numbers:: All processors have built in arithmetic for small numbers (up to 32 bits). We will use this arithmetic for such small numbers. (In fact for technical reasons the limit in GAP is $\pm 2^{28}$. There is a notable speedup if all numbers stay below 2^{28}.)

Finite field elements: Unless the field is very large, we can easily squeeze them in 4 bytes per number.

Permutations: A permutation on n points is simply stored as a list of images for each point. If $n \leq 2^{16}$ we can do with 2 bytes per point (and thus $2n$ bytes storage), in general we use 4 bytes per point and thus require $4n$ bytes of memory. (To simplify arithmetic we usually do not throw away trailing fix points. I.e. the

- The action φ of G on M/N
- If M/N is nonabelian and there is a prior chief factor $A/B \cong M/N$ the action φ of G on both chief factors simultaneously.

PERFORMANCE 168: If M/N is abelian this homomorphism φ is described easily by matrices. If M/N is nonabelian we find a small degree permutation representation for the automorphism group of its simple constituent and then (as in lemma 70) embed G^φ in a suitable wreath product.

For each image group G^φ we then test whether it can have a faithful primitive action (and classify all such actions). For this we can use the the O'Nan-Scott theorem 76.

For example if $\mathrm{Soc}(G^\varphi)$ is abelian we know that maximal subgroups must be a complement to the socle. (In fact is is not hard to see that in this case S/N must be a complement to M/N in G/N, thus rendering the action φ unnecessary.)

For the primitive actions with a non-regular normal subgroup we construct the point stabilizer S^φ by first describing $\mathrm{Stab}_{\mathrm{Soc}(G^\varphi)}(1)$ (which by theorem 76 has a very "easy" structure) and consider S^φ as its normalizer in G^φ. Then the corresponding maximal subgroup of G can be obtained as pre-image.

Some of the primitive actions are derived from permutation actions of the simple socle factor. These essentially need to be obtained from a data base, using constructive recognition (section VI.3) to connect the group to this data base (or be computed the very hard way by computing the full subgroup lattice of the simple group, which is OK if this group is comparatively small).

V.6. Intersection and Normalizer

(A section that would be here if not for time reasons.) These have been studied mainly for the case of solvable groups [GS90], a more general lifting approach for the normalizer is given in [LM02].

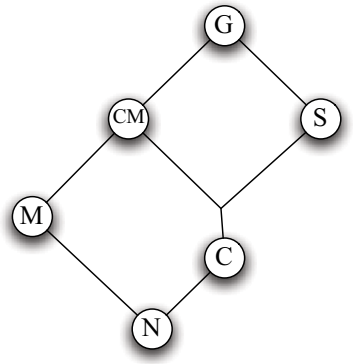


FIGURE 2. Interplay of a maximal subgroup and a chief factor M/N

LEMMA 167: Let $S < G$ be a maximal subgroup and $G = N_0 \rhd N_1 \rhd \cdots \rhd \langle 1 \rangle$ be a fixed chief series. Let φ be the action of G on the cosets of S. Then φ is isomorphic to either:

a) The action of G on one chief factor N_{i-1}/N_i or
b) The action of G on two isomorphic chief factors N_{i-1}/N_i and N_{j-1}/N_j.

Proof: The action on the cosets of S is primitive. Thus the action (by conjugation if the socle is nonabelian, affine if the socle is abelian) of G^φ on $\mathrm{Soc}(G^\varphi)$ is faithful or we can find a normal subgroups $C = \mathrm{Kern}\,\varphi \leq X \lhd G$ such that $X^\varphi = \mathrm{Soc}(G)$ and thus φ is isomorphic to the action of G on X/C.

By lemma 66 we know that $\mathrm{Soc}(G)^\varphi$ is either a) minimally normal or b) the direct product of two isomorphic minimally normal subgroups. In case a) we therefore know that $\mathrm{Soc}(G^\varphi) \cong CM/C$ and thus the action of G on CM/C is isomorphic to the action on the chief factor M/N in the given series.

If G^φ has two minimal normal subgroups they must be isomorphic. The action on M/N provides one minimal normnal subgroup. By Jordan-Hölder (the action on) the other minimal normal subgroup must be isomorphic to (the action on) another chief factor. □

We can therefore step through a chief series and for each chief factor M/N consider:

identity element of S_{10} is stored internally as images of 10 points. Internal magic makes this invisible to the user.)

Matrices: are simply lists of vectors, every vector again being a list. (GAP also uses compact types for matrices over finite fields.)

To put these numbers into context, suppose we have a permutation group acting on 1000 points. Then each permutation takes 2kB of memory. 500 permutations take 1MB. On a modern machine (2GB) we could thus store about 1 million permutations if we used up all memory. On the other hand if we have degree 100000, we could only store 500 permutations.

As we want to be able to work with such groups we clearly are only able to store a small proportion of group elements.

I.4. Orbits and Stabilizers

I.4.1. Group Actions. One of the most prominent uses of groups is symmetries of objects. Thus it should not surprise that the fundamental algorithms deal with group actions.

A group G acts on a set Ω if

- $\omega^1 = \omega$ for all $\omega \in \Omega$
- $(\omega^g)^h = \omega^{gh}$ for all $\omega \in \Omega$, $g, h \in G$.

In this case we define for $\omega \in \Omega$ the *Orbit* $\omega^G = \{\omega^g \mid g \in G\} \subset \Omega$ and the *Stabilizer* $\mathrm{Stab}_G(\omega) = \{g \in G \mid \omega^g = \omega\} \leq G$.

LEMMA 1: There is a bijection between ω^G and the right cosets $\mathrm{Stab}_G(\omega)\backslash G$ given by

$$\omega^g \leftrightarrow \mathrm{Stab}_G(\omega) \cdot g$$

In particular $\left|\omega^G\right| = \left[G{:}\,\mathrm{Stab}_G(\omega)\right]$.

If G acts on Ω, we get an homomorphism $\varphi\colon G \to S_{|\Omega|}$, we call this the *action homomorphism*.

By the properties of group actions we have that $\delta^g \in \omega^G$ for every $\delta \in \omega^G$ and every $g \in G$.

I.4.2. Computing an orbit. In general we only have generators of G, not all group elements. To calculate all images, we use the fact that every element can be expressed as product of generators and their inverses.

NOTE 2: If G is finite, we can express for $g \in G$ the inverse g^{-1} as positive exponent power of g. We therefore make the following assumption:

> If G is not know to be finite, we assume that the generating set of G contains for each generator also its inverse.

With this convention we can assume that every element of G is the product of generators of G.

The following lemma then gives the basic idea behind the orbit algorithm

LEMMA 3: Let $G = \left\langle \underline{\mathbf{g}} \right\rangle$ with $\underline{\mathbf{g}} = \{g_1, \ldots, g_m\}$ and let $\omega \in \Omega$ and $\Delta \subset \Omega$ such that

a) $\omega \in \Delta$
b) For all $\delta \in \Delta$ and every generator g_i we have that $\delta^{g_i} \in \Delta$
c) For every $\delta \in \Delta$ there exists a sequence of indices $i_1, \ldots, i_k$ such that $\delta = (\cdots(\omega^{g_{i_1}})^{g_{i_2}} \cdots)^{g_{i_k}}$

Then $\omega^G = \Delta$

Proof: By property a) and c) we have that every $\delta \in \Delta$ is in ω^G. On the other hand property b) shows that Δ must be a union of orbits. □

This gives the following algorithm:

ALGORITHM 4: The "plain vanilla" orbit algorithm.

Input: A group G, given by a generating set $\underline{\mathbf{g}} = \{g_1, \ldots, g_m\}$, acting on a domain Ω. Also a point $\omega \in \Omega$
Output: return the orbit ω^G.

```
begin
 1: Δ := [ω]
 2: for δ ∈ Δ do
 3:    for i ∈ {1,...,m} do
 4:       γ := δ^{g_i}
 5:       if γ ∉ Δ then
 6:          Append γ to Δ.
 7:       fi;
 8:    od;
 9: od;
10: return Δ
end
```

Note that the **for**-loop in line 2 runs also through elements added to *orb* in the course of the algorithm.

NOTE 5: In the algorithm, we compute the image of every orbit element under every group generator. If we do not only test whether $\gamma \in \Delta$, but identify the position of $\gamma \in \delta$ we obtain the permutation image of G. In the same way we can evaluate the action homomorphism φ.

NOTE 6: Instead of starting with ω we could start with multiple points and then calculate the union of orbits containing these points.

We now start the construction of subgroups, starting with the perfect subgroups (at least the trivial subgroup) up to F-conjugacy. In each iteration we consider the subgroups not yet processed. For every such subgroup T we form extensions $S := \langle T, x \rangle$ with x taken from the zuppos in $N_G(T)$ with $x^p \in T$ for a prime p. We also need to consider the elements x only up to $N_G(T)$ conjugacy.

We test the groups S obtained this way for duplicates and conjugates and then add them to the pool of groups to be processed on the next level.

V.4.2. Normal subgroups. For normal subgroups [Hul98] we have a similar situation, however we are looking only for normal complements. This implies that we must have a central normal subgroup and there is no B^1. We also must check whether the complements found are indeed invariant under the whole group.

(A potentially hard case is if A turns out to be elementary abelian, as there will be many complements. In such a case one can calculate submodules instead.)

To find normal subgroups of the radical factor we can use a similar approach and step along a chief series, eventually we just need to be able to find normal complements to nonabelian simple normal subgroups. These complements in fact must be centralizing, and the "dihedral group trick" (homework) can be used to find them.

V.5. Maximal Subgroups

To find maximal subgroups we use that if $S \leq G$ is maximal, the action φ of G on the cosets of S is primitive [EH01, CH04]. Let $C = \cap_{g \in G} S^g = \mathrm{Core}_G(S) = \mathrm{Kern}\,\varphi \lhd G$ be the kernel of this action.

DEFINITION 166: Let G be a group and $\varphi\colon G \to H$ and $\psi\colon G \to K$ be two epimorphisms. We that that φ is isomorphic to ψ is there is an isomorphism $\xi\colon H \to K$ such that $g^{\varphi\xi} = g^{\psi}$ for every $g \in G$.

Now suppose (figure 2) that we have a given chief series of G in which N is the largest normal subgroup such that $N \leq C$. Suppose that M is the subgroup before N in the series, i.e. M/N is minimal normal in G/N and M/N is elementary. Then $N \leq C \cap M \lhd G$ is a normal subgroup (strictly, by choice of N) contained in M, thus $C \cap M = N$. On the other hand we have that $S < MS$ (we have that $M \not\leq S$ by choice of N), the maximality of S thus implies that $G = MS$.

Therefore M/N is isomorphic to the (thus minimal) normal subgroup CM/C of G^{φ} and the action of G^{φ} on this normal subgroup is isomorphic to the action of G on M/N. We therefore get the following

the action on the complements is more complicated as in the previous section.

In practice one could first consider the factor space of H^1 corresponding to classes under the action of D with $D/M = C_{G/M}(A/M)$ and then have N act on these classes.

The algorithm for lifting then runs through all subgroups A containing M, for each such subgroup classifies the A-submodules B of M up to $N_G(A)$ conjugacy and then classifies complements up to $N_G(A) \cap N_G(B)$ conjugacy.

V.4.1. The cyclic extension algorithm. To deal with the Fitting-free factor F we use an older algorithm, called "cyclic extension" [Neu60]. Its basic idea is that a subgroup $S \leq F$ is either perfect or (using that $S' < S$) there exists a subgroup $T \lhd S$ with S/T cyclic.

Assuming that we would know T, we could find S by considering elements in $x \in N_F(T)$ and forming the extension $\langle T, x\rangle$. Clearly it suffices to consider only elements x of prime-power order (otherwise we can construct in several steps) and to consider x only up to cyclic subgroups. To this end we start the computation by determining all cyclic subgroups of prime-power order ("Zuppos" from the German term[1] for these) and for every subgroup T and its normalizer $N_F(T)$ determine the zuppos contained therein. Then the extending elements x can be chosen from among generators for the zuppos in $N_F(T)$ but not in N. (There is the issue of equal subgroups and conjugacy which we resolve by comparisons and explicit computation of conjugates of all subgroups.)

Using this approach we can construct all subgroups with the perfect subgroups (including the trivial subgroup) as seed.

To obtain the perfect subgroups let F^∞ be the terminal subgroup in the derived series of F, i.e. F/F^∞ is the largest solvable factor group.

LEMMA 165: Let $S \leq F$ be a perfect subgroup. Then $S \leq F^\infty$.

Proof: Consider $N = S \cap F^\infty$. Then S/N is isomorphic to a subgroup of F/F^∞ which is solvable. As S is perfect we have that $N = S$ and thus $S \leq F^\infty$ as claimed. □

We now can proceed in the following way to determine all perfect subgroups of F: First determine F^∞. Then, using a data base of perfect groups [HP89] and a variant of the GQuotients algorithm 102 – enforcing injectivity and not surjectivity – determine the perfect subgroups of F^∞ up to conjugacy. Test for further conjugacy under F.

[1] Zyklische Untergruppen of Primzahlpotenzordnung

PERFORMANCE 7: If we have m generators and an orbit of length n there are mn images to compute. The cost of each such image will depend on the actual action, but for this is constant.

This makes the performance-critical part of the algorithm the test $\gamma \in \Delta$ in line 6, which is essentially a search problem. A plain list storage will require $O(n)$ steps, if we can sort objects a binary search takes $O(\log n)$ steps. Techniques such as Hashing can reduce to almost constant cost per search.

An easy consequence of the orbit algorithm is that we can obtain all elements of a group G by computing the orbit of 1 under the action of G by right multiplication. In particular, we could test in an extremely crude way whether an element is in a group. (In general we want to do **much** better.)

I.4.3. Representatives. In many applications we do not only want to find the orbit of ω but also find for $\delta \in \omega^G$ an element $g \in G$ such that $\omega^g = \delta$.

We do this by calculating such elements for every orbit element. Such a list of representatives is called a *transversal*. By lemma 1 it simultaneously is a set of representatives for the cosets of $\mathrm{Stab}_G(\omega)$.

At this point it makes sense to consider ω^G as a list (with fixed ordering) to maintain a correspondence between orbit points and corresponding transversal elements.

To simplify notation, we will simply index a transversal with orbit elements: $T[\delta]$. By this we mean $T[i]$ where $\Delta[i] = \delta$. (In an implementation this merits consideration for performance purposes.)

NOTE 8: What about mapping δ to γ for arbitrary $\delta, \gamma \in \omega^G$? We simply find g, h such that $\omega^g = \delta$ and $\omega^h = \gamma$, then $\delta^{g^{-1}h} = \gamma$

For building the list of representatives we now just observe that if x is a representative for δ, then xg is a representative for δ^g. This gives the following modification of the orbit algorithm:

ALGORITHM 9: Orbit algorithm with transversal computation

Input: A group G, given by a generating set $\mathbf{g} = \{g_1, \ldots, g_m\}$, acting on a domain Ω. Also a point $\omega \in \Omega$

Output: return the orbit ω^G and a transversal T.

begin

1: $\Delta := [\omega]$
2: $T := [1]$;
3: for $\delta \in \Delta$ do
4: for $i \in \{1, \ldots, n\}$ do

```
5:      γ := δ^{g_i}
6:      if γ ∉ Δ then
7:         Append γ to Δ.
8:         Append T[δ] · g_i to T
9:      fi;
10:   od;
11: od;
12: return Δ, T
end
```

NOTE 10: It is worth observing that the representative $T[\delta]$ obtained in this algorithm is a *shortest* product of group generators that has the desired mapping. If we use the orbit algorithm to obtain all group elements, we can therefore obtain a *minimal* factorization for all group elements, however at high memory cost.

Still, if a minimal factorization is required, the only improvement to this known is to reduce the storage requirement for group elements.

I.4.4. Schreier Vectors. If you think a bit about the previous algorithm, you will notice a big problem: We store one group element for every element in the orbit. In general group elements take much more storage than orbit elements, so this is potentially catastrophic.

To avoid memory overflow, we will be using the following idea:

DEFINITION 11: Let $\Delta = \omega^G$ (again considered as a list). A *Schreier vector* (or a *factored transversal*) is a list S of length $|\Delta|$ with the following properties:

- The entries of S are generators of G (or the identity element). (In fact the entries are *pointers* to generators, thus requiring only one pointer per entry instead of one group element.)
- $S[\omega] = 1$
- If $S[\delta] = g$ and $\delta^{g^{-1}} = \gamma$ then $\gamma < \delta$ (i.e. γ precedes δ in the orbit.

We can compute a Schreier vector easily by initializing $S[\omega] = 1$. In the orbit algorithm, we then set $S[\delta] := g$ whenever a new point δ is obtained as image $\delta = \gamma^g$.

Schreier vectors can take the place of a transversal:

ALGORITHM 12: If S is a Schreier vector, the following algorithm computes for $\delta \in \omega^G$ a representative r such that $\omega^r = \delta$.

```
begin
1: γ := δ
2: r := 1
3: while γ ≠ ω do
4:    g := S[γ]
```

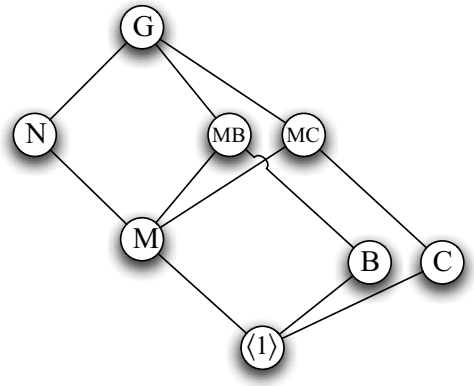



FIGURE 1. Lifting and conjugacy of complements over two steps

D centralizes X/M this fusion can be described by an action on the cohomology group describing the complements. As this action is by translation we can simply take representatives for the factor space of H^1.

V.4. Subgroups

The lifting step for subgroups is similar [Hul99, CCH01]. Assume that $M \lhd G$ is elementary abelian and that we know the subgroups of G/M (up to conjugacy). These subgroups of G/M correspond to subgroups of G containing M.

Now suppose that we have a subgroup S such that $M \not\leq S$. We set $A = \langle M, S\rangle$ and $B = S \cap M$. Then $B \lhd S$, $B \lhd M$ (as M is abelian) and thus $B \lhd A$. Thus S is a complement to M/B in A/B.

As M/B is elementary abelian we can calculate such complements in the same way as in section IV.4.1.

Next consider the action of G by conjugacy: To normalize S we clearly need to normalize A and B. Thus let $N = N_G(A) \cap N_G(B)$.

Now suppose that S and T are two complements with $S \cap M = T \cap M = B$ and $\langle S, M\rangle = \langle T, M\rangle = A$. If there is an element $g \in G$ which maps S to T this element must be in N.

We therefore need to consider complements to M/B in A/B up to N-conjugacy. Note that in general $N > A$ and $N/M \neq C_{G/M}(A/M)$. Therefore

We can therefore essentially enumerate the F-classes within S by enumerating minimal class tuples.

The classes of F outside S are relatively few, but might be small, which makes a pure random search unfeasible. Instead we can compute first the F/S classes, for each representative Sf then compute C-classes on S for $C = C_F(f)$ and finally fuse under the action of F.

V.3. Complements

We have seen already in section IV.4.1 how to compute complements to an elementary abelian normal subgroup $N \lhd G$. Here we want to consider the case of a solvable $N \lhd G$.

First we choose a chief series through N. (For example by intersecting an existing chief series with N and eliminating trivial factors.) Its factors within N will be elementary abelian, thus we need to consider a single step lifting from G/M to G with $M \lhd G$ elementary abelian.

The following theorem contains the main argument:

THEOREM 164 ([CNW90]): Suppose that $M \lhd N \lhd G$ and that $B, C \leq G$ are complements to N in G. Then:
a) MB/M and MC/M are complements to N/M in G/M.
b) B is a complement to M in MB.
c) $N_G(B) = B \cdot N_N(B)$ and $N_N(B) = C_N(B)$.
d) If B is conjugate to C in G then MB is conjugate to MC.
e) If $MB = MC$ and B is conjugate to C in G the groups are conjugate under D where $D/M = C_{N/M}(MB/M)$.

Proof: a), b) follow easily from the isomorphism theorem.
c) As $G = N \cdot B$ every element of $N_G(B)$ can be written as a product of an element $b \in B$ and an element $n \in N$. As b clearly normalizes B, n must normalizer B as well, thus $N_G(B) = B \cdot N_N(B)$. To show that $N_N(B) = C_N(B)$ consider the semidirect product $N \rtimes B$. We want an element $(n, 1)$ such that $b^n = b' \in B$. As $b^n = n^{-1}bn = \underbrace{n^{-1}n^{b^{-1}}}_{\in N} b$ this implies that $1 = n^{-1}n^{b^{-1}}$, respectively $n = n^b$.
d) is trivial
e) If $MB = MC$ the conjugating element will normalize MB. Then use c) for a description of $N_{G/M}(MB/M)$. □

Thus assume we know the complements to N/M in G/M up to conjugacy. For each complement X/M we calculate complements to M in X and then fuse the complements under D where $D/M = C_{N/M}(X/M)$. Because


```
5:    r := g · r
6:    γ = γ^{g^{-1}}
7: od;
```
end

Proof: The algorithm terminates by condition 3 for a Schreier vector. Also notice that we always have that $\gamma^r = \delta$. Thus when the algorithm terminates (which is for $\gamma = \omega$ the result r has the desired property. □

NOTE 13: In practice it makes sense to store not generators, but their inverses in the Schreier vector. This way we do not need to repeatedly invert elements. In this case r is computed by forming the product of the generator inverses *in reverse order* and inverting the final product: If $r = fgh$ then $r = (h^{-1}g^{-1}f^{-1})^{-1}$.

NOTE 14: Unless $\left|\omega^G\right|$ is very small, we will use Schreier vectors instead of a transversal and will use algorithm 12 to obtain (deterministic!) corresponding representatives. To simplify algorithm descriptions, however we will just talk about transversal elements with the understanding that a transversal element $T[\delta]$ is actually obtained by algorithm 12.

I.4.5. Stabilizer. The final modification to the orbit algorithm will let us determine a generating set for the stabilizer $\mathrm{Stab}_G(\omega)$:

LEMMA 15: (SCHREIER) Let $G = \left\langle \underline{\mathbf{g}} \right\rangle$ a finitely generated group and $S \leq G$ with $\left[G{:}S\right] < \infty$. Suppose that $\underline{\mathbf{r}} = \{r_1, \ldots, r_n\}$ is a set of representatives for the cosets of S in G, such that $r_1 = 1$. For $h \in G$ we write $\bar{h}$ to denote the representative $\bar{h} := r_i$ with $Sr_i = Sh$. Let

$$U := \{r_i g_j (\overline{r_i g_j})^{-1} \mid r_i \in \underline{\mathbf{r}}, g_j \in \underline{\mathbf{g}}\}$$

Then $S = \langle U \rangle$. The set U is called a set of *Schreier generators* for S.

Proof: As $S \cdot (r_i g_j) = S\overline{r_i g_j}$ by definition of $\bar{\cdot}$, we have that $U \subset S$.

We thus only need to show that every $x \in S$ can be written as a product of elements in U. As $x \in G = \left\langle \underline{\mathbf{g}} \right\rangle$ we can write $x = g_{i_1} g_{i_2} \cdots g_{i_m}$ with $g_{i_j} \in \underline{\mathbf{g}}$. (Again, for simplicity we assume that every element is a product of generators with no need for inverses.)

We now *rewrite* x iteratively. In this process we will define a set of elements $t_i \in \underline{\mathbf{r}}$ which are chosen from the fixed coset representatives:

$$\begin{aligned}
x &= g_{i_1}g_{i_2}\cdots g_{i_m}\\
&= t_1 g_{i_1}g_{i_2}\cdots g_{i_m} \quad \text{setting } t_1 := r_1 = 1\\
&= t_1 g_{i_1}((\overline{t_1 g_{i_1}})^{-1}\cdot \overline{t_1 g_{i_1}})g_{i_2}\cdots g_{i_m} \quad \text{insert 1}\\
&= (t_1 g_{i_1}(\overline{t_1 g_{i_1}})^{-1})t_2 g_{i_2}\cdots g_{i_m} \quad \text{where } t_2 := \overline{t_1 g_{i_1}}\\
&= \underbrace{t_1 g_{i_1}(\overline{t_1 g_{i_1}})^{-1}}_{=:u_1\in U} t_2 g_{i_2}\left((t_2 g_{i_2})^{-1}\cdot \overline{t_2 g_{i_2}}\right)\cdots g_{i_m}\\
&= u_1\cdot \underbrace{t_2 g_{i_2}\overline{t_2 g_{i_2}}^{-1}}_{=:u_2\in U}\cdot t_3 g_{i_3}\cdots g_{i_m} \quad \text{where } t_3 = \overline{t_2 g_{i_2}}\\
&\ \ \vdots\\
&= u_1 u_2\cdot\cdots\cdot u_{m-1}\cdot t_m g_m
\end{aligned}$$

Now notice that t_j is the coset representative for $g_{i_1}\cdots g_{i_{j-1}}$. Thus $\overline{t_m g_m} = 1$, as $x \in S$. Thus $t_m g_m = t_m g_m (\overline{t_m g_m})^{-1} \in U$ which gives an expression of x as product of elements in U. □

In our application we have $S = \mathrm{Stab}_G(\omega)$ and we can use a transversal as coset representatives. (The representative for the coset Sg is $T[\omega^g]$.)

We thus get the following modification to the orbit algorithm:

ALGORITHM 16: Orbit/Stabilizer algorithm

Input: A group G, given by a generating set $\underline{\mathbf{g}} = \{g_1,\ldots,g_m\}$, acting on a domain Ω. Also a point $\omega \in \Omega$

Output: return the orbit ω^G, a transversal T and the stabilizer $S = \mathrm{Stab}_G(\omega)$.

begin

1: $\Delta := [\omega]$
2: $T := [1]$;
3: $S := \langle 1\rangle$;
4: for $\delta \in \Delta$ do
5: for $i \in \{1,\ldots,n\}$ do
6: $\gamma := \delta^{g_i}$
7: if $\gamma \notin \Delta$ then
8: Append γ to Δ.
9: Append $T[\delta]\cdot g_i$ to T
10: else
11: $S := \langle S, T[\delta]\cdot g_i \cdot T[\gamma]^{-1}\rangle$

centralizers at the same time the same approach can be used in general for the computation of centralizers. Let us consider first the lifting step: We assume that $G \rhd N \cong C_p^m$ is elementary abelian and that we know class representatives (and their centralizers) in G/N.

Clearly for every $h \in G$ there exists a representative $Ng \in G/N$ such that $Nh^{Nx} = N_g$ for some $Nx \in G/N$. If g is a class representative in G we can therefore assume that $Nh = Ng$ for some class representative g of G/N. This assumption implies that we are looking for elements in Ng up to conjugacy by the subgroup C defined by $C/N = C_{G/N}(Ng)$.

Now consider this action for a fixed g. For $n \in N$, and $c \in C$ we have that $g^c = n_c g$ with $n_c = g^c/g \in N$ depending only on g and c, as $Ng = Ng^c$. Thus

$$(ng)^c = n^c g^c = n^c \cdot n_c \cdot g$$

If we consider N as a C-module, we therefore get an *affine* action: $n \mapsto n^c + n_c$ We can represent this action in matrix form (by acting on augmented vectors with 1 in their last component) and compute orbits and stabilizers this way.

Furthermore we notice that $N \lhd C$, thus the orbits of N form blocks. We want to see that we can in fact describe the action on these blocks easily: If $m \in N$ we have that $n^m = n$, thus $(ng)^m = n\cdot n_m \cdot g$ with $n_m = g^m/g$ is a pure translation. If we act by $m_1 m_2$ we translate by $n_{m_1 n_{m_2}}$, thus the translating vectors lie in a subspace $M = \langle n_m \mid m \in N\rangle \le N$. The N-orbits therefore correspond to the factor space N/M and we can consider the induced affine action of C on this factor space first. (If N is central this factor space is all we need to consider.)

V.2.1. The top step. In the first step, if G is solvable we have a trivial or an abelian group for which the problem is trivial representatives.

If G is not solvable, the following approach [Hul00] can be used to compute the classes of $F := G/O_\infty(G)$:

By a reduction to subdirect product factors we can assume without loss of generality that $T^m \le F \le \mathrm{Aut}(T)\wr S_m$. First consider the subgroup $S := F \cap \mathrm{Aut}(T)^m$. Elements of this subgroup are m-tuples of elements of $\mathrm{Aut}(T)$. Clearly we can conjugate (using T^m) in each component independently to make each entry a fixed representative of its T-class. (In fact one can relatively easily consider S-representatives of S-classes, using that $\left[\mathrm{Aut}(T){:}T\right]$ is small.)

Next one can consider the impact of component permutation on these S-classes. We can assume that a class tuple for an F-class representative in S is minimal under this action. (There also is the issue of a potential further fusion of T-classes, but this does not change the basic idea.)

- Pretabulation of results for "small" Fitting-free groups. (Up to size 10^6 there are at most a few hundered such groups and pretabulation is not that bad. The main issue is tracing errors, if mistakes are made in the pretabulated data.)
- Reductions to the simple case and special methods for these using theory. (For example we can enumerate conjugacy classes of $PSL_n(q)$ using normal forms of matrices.)

V.1.1. Factor groups. To make this approach feasible, we will have to represent the factor groups G/N on the computer.

The easiest approach is to consider the factor group as consisting of cosets and to work with elements of G as representatives and full preimages of subgroups ($U \leq G$ represents $U/N \leq G/N$). As we reduce the calculations typically to the elementary abelian factor groups themselves (which we can represent as column vector spaces) this often is sufficient.

If G/N is solvable we can compute a pcgs for G/N and represent this group as a pc group. If furthermore N (and thus G is solvable this can be done easiest by computing an IGS for N and working (as described in section IV.4.2) with "canonical" representatives for the cosets gN.

If G_{i-1}/G_i is a composition factor in the pc series of G, we have that either $G_{i-1} \leq N$ (in which case we can assume that the i-th exponent of a coset representative of N is zero) or we have that (assuming $G_{i-1} = \langle G_i, g_i \rangle$) that g_i^a and g_i^b for $a \not\equiv b \pmod{p_i}$ are in different cosets.

This shows that we can represent G/N by working with a subset of exponents.

A similar approach works if $N \leq M \lhd G$ with $N \lhd G$. In this case we compute an IGS of M and with respect to this an IGS for N, thus representing the subfactor M/N. This in particular offers an easy way to desctibe the matrix actionb of a group on an abelian chief factor.

In the nonsolvable case we can still try to compute a faithful permutation representation for G/N, for example by searching ad-hoc for subgroups containing N on whose cosets we operate. While there is no guarantee to obtain a small degree, this works reasonably well in practice. For the particular case of $G/O_\infty(G)$ or the first factor of a chief series the normal subgroup is computed as kernel of a suitable homomorphism to a permutation group of not larger degree which can be used to represent the factor group.

V.2. Conjugacy Classes

One of the earliest and easiest understandable algorithms is the computation of (representatives) of conjugacy classes of G. As we calculate


```
12:       fi;
13:    od;
14: od;
15: return Δ, T, S
end
```

PERFORMANCE 17: The set of Schreier generators has the unfortunate property to be rather large (size $\left|\underline{\mathbf{g}}\right|\ [G{:}S]$). We will see later that this is unavoidable for a certain class of groups, thus one cannot prove anything better.

On the other hand, in practice very often the set of Schreier generators is highly redundant.

We can remove obvious redundancies (duplicates, identity), but even then much redundancy remains.

There are essentially three ways to deal with this:

- For every arising Schreier generator, we test whether it is already in the subgroup generated by the previous Schreier generators. This requires many element tests.
- We just pick a small random subset of Schreier generators. We need a verification that everything went well.
- We form some random products (formally: Random subproducts) of the Schreier generators. In this case one can prove that they generate $\mathrm{Stab}_G(\omega)$ with arbitrarily high probability. Still, the verification problem remains.

I.4.6. Application: Normal Closure. Let $U \leq G$. The *normal closure* of U in G is

$$\langle U \rangle_G = \bigcap \{N \mid U \leq N \lhd G\}$$

the smallest normal subgroup of G containing U.

One of its uses is in commutator subgroups, for example if $G = \left\langle \underline{\mathbf{g}} \right\rangle$, then $G' = \left\langle a^{-1}b^{-1}ab \mid a, b \in \underline{\mathbf{g}} \right\rangle_G$.

If we start with generators of U and G, we can compute this closure in a variant of the orbit algorithm:

ALGORITHM 18: NormalClosure of a subgroup.

Input: Two generating systems $\underline{\mathbf{g}}$ and $\underline{\mathbf{u}}$ for subgroups $G = \left\langle \underline{\mathbf{g}} \right\rangle$ and $U = \langle \underline{\mathbf{u}} \rangle$.
Output: A generating system for the normal closure $\langle U \rangle_G$.

```
begin
1: n := [];
2: for x ∈ u do {start with u}
3:    Add x to n;
4: od;
```

```
 5: for d ∈ n do {orbit algorithm starting with n}
 6:   for g ∈ g do
 7:     c := d^g;
 8:     if c ∉ ⟨n⟩ then {inclusion in group closure}
 9:       Add c to n;
10:     fi;
11:   od;
12: od;
13: return n;
end
```

Proof: The algorithm clearly terminates, if G is finite, as only finitely many elements may be added to n.

As n is initialized by $\mathbf{u}$, we have that $U \leq \langle n \rangle$. Furthermore, as we only add conjugates of the elements in n, we have that $\langle n \rangle \leq \langle U \rangle_G$.

We now claim that for every $x \in \langle n \rangle$ and every $g \in G$ we have that $x^g \in \langle n \rangle$. As $(nm)^g = n^g m^g$ it is sufficient to consider $x \in n$. The statement then holds (same argument as in the orbit algorithm) as we can express g as a word in $\underline{\mathbf{g}}$.

Because of this property, we finally get that $\langle n \rangle \lhd G$ which proves that $\langle n \rangle = \langle U \rangle_G$. □

I.5. Random Elements

We have already talked (and will talk again) about using random elements. In this section we want to describe a general algorithm to form (pseudo)-random elements of a group $G = \underline{\mathbf{g}}$ if only the generating set $\underline{\mathbf{g}}$ is known.

Our first assumption is that we have a (perfect) random number generator. Using this, one can try to multiply generators together randomly. The problem is that if we only multiply with generators, the word length grows very slowly, making it difficult to obtain any kind of equal distribution in a short time.

This is resolved by multiplying products of elements together iteratively. The following algorithm is a modification of [] due to Charles Leedham-Green. It looks rather simple, but performs in practice rather well and its behaviour has been studies extensively. Unfortunately there are cases when its result will not approximate a uniform distribution.

ALGORITHM 19: (Pseudo)Random, "Product Replacement"

Let $\underline{\mathbf{g}}$ a set of group elements. This algorithm returns pseudo-random elements of $\left\langle \underline{\mathbf{g}} \right\rangle$.

CHAPTER V

Lifting

V.1. The Lifting Paradigm

Lifting is the stepwise approximation of a result in subsequently larger factor groups. As we can reach the full group is a finite number of steps, the end result will be correct. Furthermore approximation is a homomorphism which eliminates the rounding problems that plague numerical analysis.

We will assume that we have an elementary abelian normal subgroup $N \lhd G$ and that the result is known in G/N. We then want to "lift" this result to G. This means that we will have to modify elements by components in N. The aim then is to use this property to reduce the problem to a calculation in N which – N being elementary abelian – can be done using linear algebra.

By induction we can lift in several steps via a series of normal subgroups $G \rhd N_1 \rhd N_2 \rhd \cdot \rhd \langle 1 \rangle$ with $N_i \lhd G$ and N_i/N_{i+1} elementary abelian, thus effectively lifting from G/L to G if $L \lhd G$ is solvable.

In most cases, the algorithms have been initially proposed for p-groups. In this case one can assume (using a central series) that M is central, which means that we can ignore the module action.

Next, these ideas have been extended to solvable groups. Essentially this means incorporation of the group action.

In these two cases we can assume that the initial factor group is trivial, so only the lifting step has to be considered.

More recently (since about 1995) the algorithms have been generalized once more to the case of nonsolvable groups. In this situation the initial factor is $G/O_\infty(G)$. This group is "Fitting-free", i.e. it has no elementary abelian normal subgroup. Such a group is the subdirect product of groups of the form $T^m \leq G \leq \mathrm{Aut}(T) \wr S_m$ for simple groups T, where the $T_i^{m_i}$ are the socle factors. Often easy reductions to groups $G \leq \mathrm{Aut}(T) \wr S_m$ are possible. For groups of this type then the following approaches are used:

- Older algorithms. As $G/O_\infty(G)$ is smaller than G and has a more simple structure these (or just some ad-hoc approach) often succeed.

a pc-presentation for H^∞ can be used to represent very large groups and should permit generalizations of the "lifting"-type algorithms described in the next section. Currently very little is available.

The algorithm consists of an initialization step and a routine that then will return one pseudo-random group element in every iteration.

The routine keeps a (global) list X of $r = \max(11, \left|\underline{\mathbf{g}}\right|)$ group elements and one extra group element a. (Experiments show that we need $r \geq 10$, but we could change it.

Initialization:

begin

1: $X = []$;
2: $k := \left|\underline{\mathbf{g}}\right|$;
3: for $i \in [1..k]$ do
4: $\quad X[i] := g_i$;
5: od;
6: for $i \in [k+1..r]$ do
7: $\quad X[i] := X[i-k]$;
8: od;
9: $a := 1$;
10: for $i \in [1..50]$ do {50 is heuristic}
11: $\quad$ PSEUDORANDOM(); {Initial randomization}
12: od;

end

PSEUDORANDOM()

begin

1: $s :=$ RANDOM($[1..r]$);
2: $t :=$ RANDOM($[1..r] \setminus [s]$);
3: $e :=$ RANDOM($[-1, 1]$);
4: if RANDOM($[1, 2]$) $= 1$ then
5: $\quad X[s] := X[s]X[t]^e$;
6: $\quad a := aX[s]$;
7: else
8: $\quad X[s] := X[t]^e X[s]$;
9: $\quad a := X[s]a$;
10: fi;
11: **return** a;

end

(indicating M is p-elementary abelian) and $m_j h_i = h_i m_j$ (indicating M is central).

Next we check the confluence relations 156. These yield conditions among the m_i. The set of solutions yields a module M such that $M.H$ is a central extension of H. (It is the largest possible extension of this kind and is called the p-covering group of H.)

Now we also want to make the group isomorphic to a quotient of G, yielding a lift $\lambda\colon G \to E = M.H$. For this we need to evaluate the relators of G in the generator images g_i^λ. We can (this is essentially just a choice of coset representatives) assume that $g_i^\lambda = h_i$ for $i \leq k$. For the other generator images we know that $g_i^\varphi = w_i(\underline{\mathbf{h}})$ for some word w_i. We now simply set $g_i^\lambda = w_i(\underline{\mathbf{h}}) \cdot l_i$ with $l_i \in M$ another variable (which will be solved for).

Then for each relator $r(\underline{\mathbf{g}})$ we evaluate $r(\{g_i^\lambda\})$. These evaluations yield elements of M (as the relations hold in $E/M = H$). If we consider the subgroup M_1 generated by them, the factor M/M_1 is the largest central step consistent with a lift λ, thus E/M_1 is the quotient we were looking for.

This algorithm thus finds for given p and c the largest p-quotient of G of class c.

IV.5.2. Solvable Quotient: Lifting by a module. If we want not only p-groups (or finite nilpotent groups) but more generally solvable groups a couple of problems come up:

- There is not just one prime to consider.
- We don't have any longer the nice generator condition which enforces surjectivity
- M is not any longer central, but we need to consider an action of H on M.

There are essentially two approaches to this problem. One method [Nie94] generalizes the approach of the p-quotient algorithm. However it does not assume centrality. Rewriting therefore does not yield a linear system of equations, but a *module presentation*.

The second approach [Ple87] instead uses H to construct all irreducible modules for H, then forms all possible extensions with these and tests whether the existing homomorphism $\varphi\colon G \to H$ can be lifted to any of these extensions.

IV.5.3. Hybrid Quotients. Current work of the author involves a generalization to find nonsolvable quotients. Again we assume knowledge of a quotient H and want to extend to a quotient $M.H$. A confluent rewriting system is used to represent H.

The representation of such quotients H– essentially a rewriting system which is some confluent rewriting system for H/H^∞ together with

LEMMA 162: $\Phi(G)$ consists of those elements that are redundant in every generating set of G.

Proof: Homework. □

THEOREM 163 (BURNSIDE basis theorem): Let G be a finite p-group. Then $\Phi(G) = G'G^p$. If $\left[G{:}\Phi(G)\right] = p^r$, every set of generators of G has a subset of r elements, which also generates G.

Proof: Suppose $M \leq G$ maximal. Then (as p-groups are nilpotent $N_G(M)$ is strictly larger than M) we have that $M \lhd G$ and $\left[G{:}M\right] = p$ and thus $G'G^p \leq M$.

On the other hand $G/G'G^p$ is elementary abelian, and thus $\Phi(G/G'G^p) = \langle 1\rangle$. But this implies that $\Phi(G) \leq G'G^p$.

Finally let $\underline{\mathbf{g}} = \{g_1, \ldots, g_n\}$ be a generating set for G. Then $\{\Phi(G)g_i\}_{i=1}^n$ must generate $G/\Phi(G)$, which is an r-dimensional vector space. Thus we can find a subset $B = \{g_{i_1}, \ldots, g_{i_r}\} \subset \underline{\mathbf{g}}$ such that $\{\Phi(G)g_{i_1}, \ldots, \Phi(G)g_{i_r}\}$ is a basis of $G/\Phi(G)$ is a basis of $G/\Phi(G)$. But then $G = \langle B, \Phi(G)\rangle = \langle B\rangle$. □

Going back to the situation of quotient groups, now assume that we have a finitely presented group G and that $A \lhd G$ is the smallest normal subgroup such that G/A is p-elementary abelian. (We can determine this subgroup using the abelian quotient, section III.9.) Suppose that $\varphi\colon G \to H$ is a homomorphism onto a p-group H such that $\operatorname{Kern}\varphi \leq A$ (with $\alpha\colon H \to G/A$). Then $A^\varphi = \Phi(H)$ and H is generated by elements h_i^φ such that the elements h_i^α form a basis for G/A. In particular any homomorphism $\psi\colon G \to H$ for $G^\psi \cdot \Phi(H) = H$ is automatically surjective.

Suppose that we have a pc presentation for $H = \langle h_1, \ldots, h_n\rangle$ in the pc generators $\underline{\mathbf{h}}$. We assume that the first k generators in this presentation are a basis of $H/\Phi(H)$. We also assume that for each $j > k$ there is one relation in which h_j occurs which is labeled as *definition* of h_j (in terms or earlier generators, thus penultimately in terms of $h_1, \ldots, h_k$), the other relations are called *non-defining* relations.

We also assume that $h_i = g_i^\varphi$ for $i \leq k$. This is up to a reordering of $\underline{\mathbf{g}}$ without loss of generality, as $h_1, \ldots, h_k$ are a basis of $H/\Phi(H)$ and we simply form this basis by choosing subsequent irredundant images $\Phi(H) \cdot g_i^\varphi$.

To form a larger p-group image E we replace every non-defining relation $w(\underline{\mathbf{h}})$ by a relation $w(\underline{\mathbf{h}})m_i$ ($i = 1, 2, \ldots$) with m_i an element of the presumptive module $M = \langle \underline{\mathbf{m}}\rangle$. We also add relations $m_i m_j = m_j m_i$, $m_i^p = 1$

CHAPTER II

Permutation Groups

Probably the most important class of groups are permutation groups, not least because every finite group can be represented this way. If you are interested in details, there is a monograph [Ser03] dedicated to algorithms for such groups which goes in much more detail.

II.1. The Schreier-Sims algorithm

We now assume that G is a (potentially large) permutation group, given by a set of permutation generators. We want to compute with this group (for example: find its order, and to have an element test), without having to enumerate (and store!) all its elements. Obviously we have to store the generators, we also are willing to store some further group elements, but in total we want to store just a few hundred elements, even if the group has size several fantastillions.

II.1.1. Stabilizer Chains. The algorithm we want to develop is due to Charles Sims [Sim70], as it uses Schreier's lemma 15 this algorithm has been known commonly as the "Schreier-Sims" algorithm.

Its basic idea is the following: We consider a list of points $B = (\beta_1, \ldots, \beta_m)$, such that the (point-wise) stabilizer of B in G is trivial. We call such a set B a *base* for G. Corresponding to the base we get a *Stabilizer chain*: This is a sequence of subgroups of G, defined by $G^{(0)} := G$, $G^{(i)} := \operatorname{Stab}_{G^{(i-1)}}(\beta_i)$. (By the definition of a base, we have that $G^{(m)} = \langle 1\rangle$.)

One interesting property of a base is that every permutation $g \in G$ is determined uniquely by the images of a base $\beta_1^g, \ldots, \beta_m^g$ it produces. (If h produces the same images, g/h fixes all base points.)

NOTE 20: In general a base is rather short (often length < 10 even for large groups) but there are obvious cases (e.g. symmetric and alternating groups) where the base is longer. Still as every stabilizer index must be at least 2, the length of a base must be bounded by $\log_2 |G|$.

Sims' idea now is that we can describe G in terms of the cosets for steps in this chain: An element $g \in G^{(i-1)}$ will be in a coset of $G^{(i)}$. Thus we have that $g = a \cdot r$ with $a \in G^{(i)}$ and b a coset representative for $G^{(i)}$ in

$G^{(i-1)}$. As $G^{(i)} = \mathrm{Stab}_{G^{(i-1)}}(\beta_i)$ these coset representatives correspond to the orbit of β_i under $G^{(i-1)}$.

By using this kind of decomposition inductively, we can write any $g \in G$ in the form $g = b_m b_{m-1} \cdots b_1$ with b_i a coset representative for $G^{(i)}$ in $G^{(i-1)}$ and thus corresponding to a point in the orbit $\beta_i^{G^{(i-1)}}$.

We can describe these orbits and sets of representatives using the orbit algorithm we studies in the last chapter.

On the computer we thus store a stabilizer chain in the following way:

Each subgroup $G^{(i)}$ in the stabilizer chain is represented by a record with entries giving

- the `generators` of $G^{(i)}$,
- the `orbit` of β_{i+1} under $G^{(i)}$ (we shall use the convention that $\beta_{i+1} =$ `orbit`[1]),
- a corresponding `transversal` (which in fact will be implemented using a Schreier vector) and
- a pointer to the `stabilizer` which is the record for $\mathrm{Stab}_{G^{(i)}}(\beta_{i+1})$.

EXAMPLE 21: Let $G = A_4$ with base $(1,2)$. Then $G = G^{(0)} = \langle (1,2,3), (2,3,4) \rangle$, $G^{(1)} = \mathrm{Stab}_G(1) = \langle (2,3,4) \rangle$ and $G^{(2)} = \mathrm{Stab}_G(1,2) = \langle \rangle$.

We thus get (for example) the following data structure:

```
rec(generators:=[(1,2,3),(2,3,4)],
    orbit:=[1,2,3,4],
    transversal:=[(),(1,2,3),(1,3,2),(1,4,2)],
    stabilizer := rec(
       generators:=[(2,3,4)],
       orbit:=[2,3,4],
       transversal:=[(),(2,3,4),(2,4,3)],
       stabilizer:= rec(
         generators:=[])  )  )
```

NOTE 22: How do we actually determine the base? We simply determine the next base point when we need it: β_i is simply chosen to be a point moved (so we have a proper orbit) by some generator of $G^{(i-1)}$.

In some applications, one also might want a base containing particular points, which we would chose first.

A naive way to calculate a stabilizer chain would be to simply compute the orbit of β_1 under $G = G^{(0)}$ and generators for $G^{(1)} = \mathrm{Stab}_{G^{(0)}}(\beta_1)$ using the Orbit/Stabilizer algorithm. We then iterate for $G^{(1)}$ until we end up with a trivial stabilizer.

The only problem with this approach is the large number of Schreier generators: In each level the number of generators will increase by the index, leaving us about $|G|$ generators in the last step.

Examples of varieties are solvable groups, nilpotent groups, p-groups or abelian groups. Furthermore one could impose for example conditions on the length of certain "natural" normal series.

LEMMA 159: Let G be a group. For every variety $\mathcal{V}$ there is a smallest normal subgroup $N_{\mathcal{V}} \lhd G$ such that $G/N_{\mathcal{V}}$ is in $\mathcal{V}$.

Proof: If $N, M \lhd G$ both have the property, then $G/(N \cap M)$ is a subdirect product of G/N with G/M. □

The quotient algorithms we will study aim to construct for a finitely presented group G the largest quotient group $F = G/N$ in a certain variety, possibly subject to conditions of order or length of a composition series. (The lemma shows that this is a sensible aim to have.)

NOTE 160: We could apply such an algorithm in particular to free groups. This gives an – initially crude – way of constructing all groups of a given order in a particular variety. The difficulty however is the elimination of isomorphic groups.

This approach has been refined for particular cases, incorporating a reasonably efficient rejection of isomorphic duplicates. The resulting algorithms are the p-group generation algorithm [O'B90] and the "Frattini extension" algorithm to construct all solvable groups of a given order [BE99].

The idea behind the quotient algorithms is as follows:

Assume that we know already a homomorphism $\varphi\colon G \to H$ which represents a smaller quotient (often the largest quotient in which the chief series length is bounded by one less).

We want to find a larger quotient $\lambda\colon G \to E$ such that $\mathrm{Kern}\,\lambda < \mathrm{Kern}\,\varphi$ and $M = \mathrm{Kern}\,\varphi / \mathrm{Kern}\,\lambda$ is elementary abelian (if M was solvable, we could compose it from elementary abelian groups in several steps). We thus have that E must be an extension of M by H.

We create a formal extension of this group M with H. This means, a relation $l \to r$ in H becomes $l \to r \cdot m$ in $E = M.H$. We consider these m_i as variables and then use confluence (respectively the consistency conditions for pc groups) to obtain conditions on the values of these.

IV.5.1. p-Quotient. Let us first consider the case of the quotients being p-groups (finite nilpotent groups, being the direct product of p-groups are essentially equivalent).

We need a bit of theory about irredundant generating systems:

DEFINITION 161: Let G be a finite group. The *Frattini-subgroup* $\Phi(G) \leq G$ is the intersection of all maximal subgroups of G.

series. A particular nice situation is that of so-called "special pc-groups" (defined by a series of rather technical conditions [CELG04]), which not only provide particularly good algorithms, but also often a very "sparse" presentation which makes the collection process go fast.

Sometimes it is possible to immediately write down the pc presentation for a group of interest (for example, it is not hard to construct the pc-presentation for a semidirect product from pc-presentations for the factors).

If we have a solvable group given already as group of permutations or matrices, we could obtain a pc-presentation as in section III.10.2. There is however a more efficient algorithm [Sim90].

A third main source is as the results of certain quotient algorithms which we will study below.

Induced pc systems. Suppose that G is a pc group and $S \leq G$. Suppose we have some generating set $\underline{\mathbf{s}} = \{s_1, \ldots, s_k\}$ for S. Consider the exponent vectors for the s_i as rows in a matrix.

Suppose that s_i corresponds to an exponent vector $[a, \ldots]$. Then a power s_i^e will have exponent vector $[1, \ldots]$ (as $\gcd(a, p) = 1$ for p being the relative order in the first component).

Similarly, if s_j has coefficient vector $[b, \ldots]$, then $s_j / s_i^{b/a}$ has exponent vector $[0, \ldots]$.

It is easily seen that these transformations do not change the group generated.

We can therefore transform $\underline{\mathbf{s}}$ to a new generating set $\underline{\hat{\mathbf{s}}}$ such that the matrix of exponent vectors is in row echelon form. Such a generating set $\underline{\hat{\mathbf{s}}}$ is called an *induced generating set* (or *IGS*) for S. If we assume reduced row echelon form, we can even obtain a unique generating set for S, called a *canonical generating set* (or *CGS*).

If we have an induced generating set for S and $g \in G$ arbitrary, we can attempt to divide generators of the IGS off g to transform the exponent vector for g in the zero vector. This will succeed if and only if $g \in S$. The remainder will be a "canonical" (with respect to the chosen IGS) coset representative for the *left coset* gS. (We have to divide off from the right to avoid commutators with remaining higher terms causing problems.)

IV.5. Quotient Algorithms

As an application of pc presentations we consider again the problem of finding quotients of a finitely presented group.

DEFINITION 158: A *variety* of groups is a class of groups that is closed under subgroups, factor groups and direct products

II.1.2. Element Test. The remedy for this problem lies in the fact that a stabilizer chain lets us to do an element test for the group represented by the chain:

We do this by trying to write the element in question as a product of transversal elements:

ALGORITHM 23: Let $g \in G^{(0)}$. We want to find the expression $g = b_m b_{m-1} \cdots b_1$ with b_i a coset representative for $G^{(i)}$ in $G^{(i-1)}$.

Input: A stabilizer chain C for a group G and an element $g \in G$

Output: A list $L = [b_1, b_2, \ldots, b_m]$ of coset representatives, such that $g = b_m b_{m-1} \cdots b_1$.

begin

1: $L :=$ [];
2: while C.`generators` <> [] do
3: $\beta := C$.`orbit`[1];
4: $\delta = \beta^g$;
5: $r := C$.`transversal`[δ];
6: $g := g/r$;
7: Add r to L;
8: $C := C$.`stabilizer`;
9: od;
10: **return** L

end

Proof: Observe that $\beta^r = \beta^g$, thus the new g in line 6 is in the stabilizer of β. Thus at the end of the algorithm, after dividing off representatives, we must have $g = 1$. □

Now consider what happens if $g \notin G$. Then obviously the algorithm cannot terminate with $g = 1$. Instead two things can happen:

- δ may not be in the orbit of β in some iteration of the loop.
- At the end of the algorithm we have $g \neq 1$.

If we test for these two properties, we get a constructive element test. We call this resulting procedure "ElementTest(C, a)". This process also is sometimes called "*sifting*".

II.1.3. The Schreier-Sims algorithm. This element test gives us the chance to remove redundant Schreier generators: We will build the stabilizer chain not layer by layer, accumulating a large number of Schreier generators, but instead after obtaining one Schreier generator start (or continue) building the next level.

In doing so, however we test first whether the new generator is recognized by the element test 23. If it is, we can ignore it.

To use this element test, we will have to assume that below the level in which we are working (i.e. the C.`stabilizer` component) is a correct chain for its given `.generators`.

We can ensure this condition holds, by only ascending one level once all Schreier generators have been tested.

Whenever we pass a subsequent Schreier generator down the chain, we also will have to deal with an already existing (partial) chain. If the new generator does not pass the element test, and we have to extend the chain we can avoid obvious repetition of images (old points under old generators) in the orbit algorithm and similar obvious repetition of Schreier generators that must have been tested already.

The resulting algorithm is recursive. In the version presented here, the algorithm picks base points itself, though one can obviously "seed" a partial base.

ALGORITHM 24: Recursive version of the Schreier-Sims algorithm. As the main algorithm is a recursive function (EXTEND), we need to perform a separate initialization.

Input: A generating set $\underline{\mathbf{g}}$ for a permutation group G
Output: A recursive data structure for a stabilizer chain for G.
begin
1: $C :=$ `rec(generators :=` []);
2: for $a \in \underline{\mathbf{g}}$ do
3: EXTEND(C, a);
4: od;
5: **return** C;
end

The actual work is done in the following recursive function which extends and modifies the chain C.
EXTEND(C, a)
begin
1: if ElementTest(C, a) fails then {Extend existing stabilizer chain}
2: if C.`generators` $=$ [] then {We are on the bottom of the chain}
3: C.`stabilizer` := `rec(generators :=` []); {Add a new layer}
4: $\beta :=$ one point moved by a; {or a predefined next base point}
5: Add a to C.`generators`;
6: C.`orbit` := $[\beta]$;C.`transversal` := [1];
7: $\delta := \beta^a$; $s := a$;{Special orbit algorithm for single generator}
8: while $\delta \neq \beta$ do
9: Add δ to C.`orbit`; Add s to C.`transversal`;
10: $\delta := \delta^a$; $s := s \cdot a$;
11: od;

COROLLARY 156 ([Wam74][3]): A presentation of a form as given in section III.10.2 with $2 \leq p_i < \infty$ for every i yields a confluent rewriting system, if the following conditions hold:

Overlap	Reduction 1		Reduction2	
$g_i^{p_i+1} =$	$g_i v_i$	$=$	$v_i g_i$	
$g_j^{p_j} g_i =$	$v_j g_i$	$=$	$g_j^{p_j-1} g_i w_{i,j}$	$(i < j)$
$g_j g_i^{p_i} =$	$g_j v_i$	$=$	$g_i w_{i,j} g_i^{p_i-1}$	$(i < j)$
$g_k g_j g_i =$	$g_j w_{j,k} g_i$	$=$	$g_k g_i w_{i,j}$	$(i < j < k)$

(There are generalizations for infinite polycyclic groups.)

Proof: These are all possible overlaps. □

PERFORMANCE 157: One can show that a subset of such conditions suffices.

This test makes it possible to use a polycyclic presentation to *define* a solvable group on the computer. (Such groups are called `PcGroups` in GAP.)

We can keep elements as (normal form) words $g_1^{e_1} \cdots g_n^{e_n}$ with $0 \leq e_i < p_i$ for every i. Thus we only need to store the *exponent vector* $[e_1, e_2, \ldots, e_n]$.

Arithmetic for such groups is done by concatenation, followed by collection to normal form.

Such groups have a couple of algorithmically interesting properties:

- The storage of elements is very space efficient.
- The natural homomorphism for the factor groups related to the composition series is easily evaluated (trimming exponent vectors)
- For chief factors of the form C_p^k in the series, the corresponding part of the exponent vector yields a vector space representation. This is the principal idea behind many efficient algorithms (see chapter V).
  For this reason we might want to have the possibility to compute exponent vectors also for solvable groups which are not represented by words. In this case we call the set of generators $g_1, \ldots, g_n$ a *polycyclic generating set* (short *PCGS*) and assume an underlying data structure — for example based on a stabilizer chain — which provides a means for computing exponents.

In general, there are multiple conditions one would like the corresponding pc-series to fulfill. For example one might want it to refine a chief

[3] Originally proven directly for nilpotent groups without recuse to rewriting systems

EXAMPLE 155: Consider $G = S_4 = \langle a = (1,2,3,4), b = (1,2)\rangle$ and the normal subgroup $N = \langle (1,2)(3,4), (1,3)(2,4)\rangle \lhd G$. Then $a \mapsto (1,3)$, $b \mapsto (1,2)$ is a homomorphism with kernel N. The action of G on N is described by the matrices $a \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $b \mapsto \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

The factor group $G/N \cong S_3$ has (in these images) the presentation

$$\langle x, y \mid x^2 = y^2 = (xy)^3 = 1\rangle .$$

We now want to find elements of the form an, bm (with $n, m \in N$) which fulfill these relations. We get the following equations:

$$\begin{aligned} x^2 : (an)^2 &= a^2 n^a n = a^2 n^{a+1} \\ y^2 : (bm)^2 &= b^2 m^b m = b^2 m^{b+1} \\ (xy)^3 : (anbm)^3 &= (ab)^3 n^{babab} m^{abab} n^{bab} m^{ab} n^b m = (ab)^3 n^{babab+bab+b} m^{abab+ab+1} \end{aligned}$$

We now assume N as a 2-dimensional vector space over $\mathbb{F}_2$ with basis as given. Thus $n = [n_1, n_2]$ and $m = [m_1, m_2]$. We also evaluate the expressions $a^2 = (1,3)(2,4) = [0,1]$, $b^2 = () = [0,0]$, $(ab)^3 = () = [0,0]$. The equations thus become in vector form:

$$\begin{aligned} -a^2 = [0,1] &= n^{a+1} = [n_1, n_2] \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = [0, n_1] \\ -b^2 = [0,0] &= m^{b+1} = [m_1, m_2] \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = [m_2, 0] \\ -(ab)^3 = [0,0] &= n^{babab+bab+b} m^{abab+ab+1} \\ &= [n_1, n_2] \cdot \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + [m_1, m_2] \cdot \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = [0,0] \end{aligned}$$

which yields the following system of (nontrivial) equations:

$$\begin{aligned} n_1 &= 1 \\ m_2 &= 0 \end{aligned}$$

whose solutions correspond to complements. For example the solution $n_1 = m_1 = 1$, $n_2 = m_2 = 0$ corresponds to the generators

$$(1,2,3,4) \cdot (1,2)(3,4) = (2,4) \qquad \text{and} \qquad (1,2) \cdot (1,2)(3,4) = (3,4).$$

IV.4.2. Polycyclic Presentations. Let us now return to the pc presentations for solvable groups, which we studied in section III.10.2:

We have power relations $g_i^{p_i} = v_i(g_{i+1}, \ldots, g_n)$. We consider the conjugation rules (for $j > i$) of the form $g_j^{g_i} = w_{i,j}(g_j, \ldots, g_n)$ as rewriting rules $g_j g_i = g_i w_{i,j}(g_j, \ldots, g_n)$ with respect to an (iterated) wreath product ordering.

The confluence condition 143 yields the following easy consequence


```
12:     EXTEND(C.stabilizer, s);{s is only Schreier generator}
13:   else {The layer already has an existing orbit}
14:     O := C.orbit; T := C.transversal; {Extend orbit algorithm}
15:     l := |O|;
16:     for δ ∈ O in position 1 to l do {Old points only with new generator}
17:       γ = δ^a;
18:       if γ ∉ O then
19:         Add γ to O; update transversal;
20:       else
21:         s := T[δ]aT[γ]^{-1};
22:         EXTEND(C.stabilizer, s);
23:       fi;
24:     od;
25:     for δ ∈ O in position > l do {new points with all generators}
26:       for b ∈ C.generators ∪ {a} do
27:         γ = δ^b;
28:         if γ ∉ O then
29:           Add γ to O; update transversal;
30:         else
31:           s := T[δ]bT[γ]^{-1};
32:           EXTEND(C.stabilizer, s);
33:         fi;
34:       od;
35:     od;
36:     Add a to C.generators;
37:   fi;
38: fi;
end
```

PERFORMANCE 25: We only process a if the element test in line 1 fails. In this case the test will fail on some (potentially lower) level in the chain after already dividing off transversal elements on a higher level. As this "sifted" element is moving fewer points it is preferably taken in place of a. This way it will give immediately Schreier generators on a lower level.

NOTE 26: If G is known to be solvable, there is a better, algorithm that has been proposed by Sims in 1990 [Sim90].

II.1.4. Strong Generators. The reason for the recursive structure of the Schreier-Sims algorithm is that we do not know immediately a reasonable set of generators for the different stabilizers. If we did, we could build the stabilizer chain very quickly layer by layer, just using the orbit algorithm. This motivates the following definition:

DEFINITION 27: A *Strong generating system* (SGS) for G is a generating set S for G, such that the i-th stabilizer $G^{(i)}$ is generated by $S \cap G^{(i)}$.

If we have a stabilizer chain, the union of the `generators` components on all levels obviously yields a strong generating system.

Given a strong generating set, we can thus very easily rebuild a stabilizer chain. This explains, why the computation of a stabilizer chain is often described as computation of a base and a strong generating system.

PERFORMANCE 28: A small problem in the construction of a Schreier vector is that the algorithm may produce unwieldy large expressions for representatives. Consider for example the group

$$G = \langle a = (1,2,3,4,\ldots,100), b = (1,2)\rangle\,.$$

With this generating set, the representative for i will be a^i, respectively for $i > 50$ the power $a^{-(101-i)}$. On the other hand, as $G = S_{100}$, there are other generating sets, which produce in average much shorter representative words.

This problem is magnified by the fact that we iteratively add generators.

A way around this problem is to add further group elements (short products) to the generating set.

In particular, one could rebuild the stabilizer chain with a strong generating set as generators to obtain immediately multiple generators on each level.

II.1.5. Base images and Permutation words. The most expensive subtask of the Schreier-Sims algorithm is the multiplication of permutations, in particular if we have to get transversal elements from a Schreier vector. To improve performance, it is thus desirable to reduce the number of multiplications.

There are two approaches to do this:

Base Images: If we already know a base $B = (\beta_1,\ldots,\beta_m)$, we know that every permutation g is determined uniquely by the *base image* $(\beta_1^g,\ldots,\beta_m^g)$ it produces.

Now suppose that $(\gamma_1,\ldots,\gamma_m)$ is a base image under some group element g and we have $h \in G$. Then $(\gamma_1^h,\ldots,\gamma_m^h)$ is the base image for gh.

We thus can represent group elements in the algorithm by their base images. The cost of one multiplication then is proportional to the length of a base and not, as permutation multiplication would be, to the length of the domain.

This is in particular relevant if we work with a subgroup $U \le G$ and have already a base for G determined.

consider the case of $N \cong \mathbb{F}_p^m$ elementary abelian. The case of a solvable N then can be dealt with by lifting methods as described in chapter V.

NOTE 153: As classes of complements are parameterized by the 1-Cohomology group this process can also be considered as a way of calculating 1-Cohomology groups.

Suppose that $G/N = \langle Ng_1,\ldots Ng_k\rangle$ and that C is a complement to N. Then we can find elements $c_i \in C$ such that $C = \langle c_1,\ldots c_k\rangle$ and $Ng_i = Nc_i$. The map $G/N \to C$, $Ng_i \mapsto c_i$ is a homomorphism.

Vice versa every homomorphism $\varphi\colon G/N \to G$, such that $\langle N, G/N^\varphi\rangle = G$ must be (as $|G/N| = |G/N|^\varphi$) a monomorphism and define a complement G/N^φ to N in G.

The task of finding a complement therefore is equivalent to the task of finding elements $c_i \in G$ such that the map $G/N \to G$, $Ng_i \mapsto c_i$ is a homomorphism and that $g_i/c_i = n_i \in N$.

We do this by considering the $n_i \in N$ as variables. We want to come up with a system of equations, whose solutions correspond to complements (and unsolvability implies that no complements exist).

For this, suppose that we have a presentation for G/N in the elements Ng_i, say $\langle \underline{\mathbf{f}} \mid r(\underline{\mathbf{f}}) = 1, r \in R\rangle$. (In practice one would calculate a presentation and then choose the generators accordingly.) We therefore want that

$$1 = r(g_1n_1,\ldots,g_kn_k). \tag{154}$$

We now rewriting these relators with rules reflecting the extension structure: $ng = gn'$ with $n,n' \in N$. As $n' = n^g$ this can be described by the action of G/N on N. We also have that $n_i^g n_j^h = n_j^h n_i^g$ because N is abelian.

With these rules we can collect the terms g_i to the left in the same order as in the original relator. Equation (154) thus becomes

$$1 = r(g_1,g_2,\ldots,g_k)\prod n_i^{w(\underline{\mathbf{g}})}$$

where the $w(\underline{\mathbf{g}})$ is a word in the group algebra $\mathbb{F}_pG/N$.

For example, if $r = f_1f_3f_2f_3$, we rewrite as

$$1 = g_1g_3g_2g_3\cdot n_1^{g_3g_2g_3}n_3^{g_2g_3}n_2^{g_3}g_3 = g_1g_3g_2g_3\cdot n_1^{g_3g_2g_3}n_2^{g_3}g_3^{g_2g_3+1}$$

In this expression we can explicitly evaluate $r(g_1,\ldots,g_n) \in N$ (which will give the (inverse of) the right hand side of linear equations. If we consider each $n_i = (n_{i,1},\ldots,n_{i,m})$ as a column vector with variable entries, the remaining part $\prod n_i^{w(\underline{\mathbf{g}})}$ yields linear equations in the variables $n_{i,j}$.

Considering all relators thus gives an inhomogeneous system of equations, whose solutions describe complements.

are reduced. Multiplication of elements then consists of concatenation and subsequent reduction.

In GAP, one can enforce such a reduction for a given finitely presented group with the command `SetReducedMultiplication(G);`

DEFINITION 148: This process of reduction to normal form is called *collection*[2].

PERFORMANCE 149: While confluence implies that the order of applying reductions does not have an impact on the final reduced (normal) form, it can have a substantial impact on the runtime. This question has been studied primarily for the case of pc presentations (see below).

IV.4. Rewriting Systems for Extensions

Similar to the situation for presentations (section III.10.2), we want to construct a rewriting system for a group from rewriting systems for a normal subgroup and for its factor group.

The key to this is to combine two orderings on two alphabets to the so-called wreath product ordering on the union of alphabets:

DEFINITION 150: Suppose that $\prec_A$ is an ordering on an alphabet A and $\prec_B$ and ordering on an alphabet B. We consider the disjoint union $A \cup B$. On this set we define the *wreath product ordering* $\prec_A \wr \prec_B$ as follows:

We can write a word in $A \cup B$ as a product of words in A and in B: Let $v = a_0 b_1 a_1 b_2 a_2 \dots a_{m-1} b_m a_m$ and $w = c_0 d_1 c_1 d_2 c_2 \dots c_{n-1} d_n c_n$ with $a_i, c_i \in A^*$ and only a_1 or a_m permitted to be the empty word (and ditto for c) and $b_i, d_i \in B$ or empty. Then $v \prec_A \wr \prec_B w$ if $b_1 b_2 \dots b_m \prec_B d_1 d_2 \dots d_n$, or if $b_1 b_2 \dots b_m = d_1 d_2 \dots d_n$ and $[a_0, a_1, \dots, a_m]$ is smaller than $[c_0, c_1, \dots, c_n]$ in a lexicographic comparison, based on $\prec_A$.

LEMMA 151: If $\prec_A$ and $\prec B$ are reduction orderings, then $\prec_A \wr \prec B$ is.

NOTE 152: A rule $ab \to ba'$ with $a, a' \in A^*$ and $b \in B^*$ is reducing with respect to $\prec_A \wr \prec_B$. Thus (A representing a normal subgroup and B representing a factor group) wreath product orderings permit to combine rewriting systems of a group from a rewriting system for a factor group and its normal subgroup.

IV.4.1. Complements. If $N \lhd G$ we define as *complement* to N a subgroup $C \leq G$ such that $N \cap C = \langle 1 \rangle$ and $G = NC$. (In other words: $G \cong N \rtimes C$).

Given G and $N \lhd G$ we want to find whether such a complement exists (and later want to consider complements up to conjugacy). Here we will

[2]The name stems from the special case of polycyclic presentations for p-groups, for which this process has been studied in a purely theoretical context in [Hal33]

Words: Instead of multiplying out permutations, we can store products as a *word* of permutations, i.e. fgh is stored as $[f, g, h]$. Multiplication of words is simple concatenation; the inverse of $[f, g, h]$ is $[h^{-1}, g^{-1}, f^{-1}]$; the image of a point ω under $[f, g, h]$ can be computed as $((\omega^f)^g)^h$. The only test which is hard is to determine whether a word represents the identity, as we need to compute the images of all points (unless we know a base).

II.1.6. Randomization. The biggest problem with the Schreier-Sims algorithm is the large number of Schreier generators on each level – the problem is that we have no criterion which elements we can safely ignore.

Experiments show that one can usually ignore at least half the generators, but there are more problematic cases. This can be rectified, to give a statistically satisfactory behavior, but is a rather complicated process.

Another way to look at this is that if we only pick some Schreier generators, we effectively rebuild a stabilizer chain with a set S as strong generating set, which is in effect a proper subset of a strong generating set. Consequentially the chain is not describing the group but a proper subset. As every proper subgroup has index 2 one would thus expect that the element test with this chain for a random group element will fail with probability $\frac{1}{2}$. Indeed this is true as the following lemma shows:

LEMMA 29: Suppose we have built a stabilizer chain for a group G which is missing Schreier generators on some level (and thus has too short orbits on some levels). Then an element of G fails the element test for this chain with probability at least $\frac{1}{2}$.

Proof: Let $S^{(j)}$ be the groups generated by the Schreier generators on the respective level of the chain and $G^{(j)}$ the proper stabilizers. Let i be the largest index in the stabilizer chain, such that $S^{(i+1)} \neq \mathrm{Stab}_{S^{(i)}}(\beta_i)$. Then $S^{(i+1)}$ in fact has a correct chain (otherwise i was larger) and we can do a proper element test in this group.

Now consider the element test with the given chain S for group elements. Suppose that the probability is p, that uniformly random elements $g \in G$ sift through levels 1 to i. Let $\bar{g}$ be the product of transversal elements divided off at this point. Then $r = g/\bar{g} \in G^{(i+1)}$. Furthermore (multiply one element that passes with elements of $G^{(i+1)}$) every element of $G^{(i+1)}$ occurs as r with the same probability.

On the other hand, as the chain is wrong, we know that $S^{(i+1)} \neq G^{(i+1)}$, thus $\left[G^{(i+1)}{:}S^{(i+1)}\right] \geq 2$. Thus r passes the element test for $S^{(i+1)}$ with probability $\leq \frac{1}{2}$. Sifting thus fails at least with probability

$$(1-p) + p\frac{1}{2} = 1 - \frac{p}{2} \geq \frac{1}{2}$$

□

If we suppose that generators passed to the next level of the Schreier-Sims algorithm are uniformly distributed (which is not true, but often not too wrong), we can thus take the passing of the element test to indicate with probability $\geq \frac{1}{2}$ that the chain is in fact correct. If subsequent Schreier generators do not extend the chain, this probability grows. One thus could stop processing further Schreier generators, once a fixed number of Schreier generators in a row did not extend the chain.

Furthermore, in the "Random Schreier-Sims" algorithm as proposed in [Leo80], we can form (Pseudo-)random elements of the group G (e.g. using algorithm 19) and test whether a fixed number (20 elements is used in [Leo80]) of them pass the element test with chain.

II.1.7. Verification. The "only" problem with even the best randomized approach is that we can never guarantee that we obtained a correct stabilizer chain. If (a rhetorical if as a mathematician) we want to obtain proven results, we need to *verify* the obtained chain.

The following methods can be used for such a verification:

Known Order: The only way a chain can be incorrect if we are missing generators on some level, and consequently an orbit on this level or lower becomes to short. In this situation the order of the group as calculated from the stabilizer chain is too small. If we know $|G|$ we can simply compare.

Combinatorial verification: Charles Sims developed in 1970 an combinatorial algorithm for verifying a stabilizer chain obtained with random methods but did not publish the method. The first description can be found in [Ser03].

Presentations: One can use the stabilizer chain to deduce "rules" which have to hold in the group – if the chain is too small some will fail. This will lead to a so-called Todd-Coxeter-Schreier-Sims algorithm.

Using a Composition series: If we know a composition series, we can verify all composition factors separately, see III.10.4

If the verification of a chain fails, we have to continue adding Schreier generators. (Often the tests provide already particular elements that should be used.

II.1.8. Changing the base. In some situations it is desirable to have a stabilizer chain for a particular base. We can certainly achieve this by building a new stabilizer chain. If a chain already exists, we know the order of the group, and thus can safely use a randomized approach.

begin

```
 1: pairs := [];
 2: for p ∈ L do
 3:    for q ∈ L do
 4:       Add (p, q) and (q, p) to pairs.
 5:    od;
 6: od;
 7: while |pairs| > 0 do
 8:    remove a pair (p, q) from pairs.
 9:    for all overlaps w = xabc with left(p) = ab and left(q) = xbc and (x = ∅
       or a = ∅) do
10:       Let w_p = x · right(p) · c and w_q = a · right(q).
11:       Using L, reduce w_p to a reduced form z_p and w_q to z_q.
12:       if z_p ≠ z_q then
13:          if z_p ≺ z_q then
14:             Let r be a new rule z_q → z_p.
15:          else
16:             Let r be a new rule z_p → z_q.
17:          fi;
18:          Add r to L.
19:          for p ∈ L do
20:             Add (p, r) and (r, p) to pairs.
21:          od;
22:       fi;
23:    od;
24: od;
```

end

Proof: We are testing explicitly the conditions of lemma 143 for all pairs. □

NOTE 146: Analogous to Gröbner bases, one can define a "reduced" confluent rewriting system in which no left hand side can be reduced by the remaining rules.

Analogous to theorem 111 we remark

THEOREM 147: If $\mathcal{R}$ is a rewriting system describing the finite group G, then the Knuth-Bendix algorithm will terminate after finite time with a confluent rewriting system.

IV.3.1. Arithmetic: Collection. Once we have a confluent rewriting system for a group, we can compute the normal form for any word, and thus compare elements. Typically one would simply assume that all words

Proof: Suppose the two reduction are $d \to e$ and $f \to g$. If d and f occur in w without overlap, we can still apply both reductions in either order and obtain the same result q:

$$\begin{array}{lcccccc} q = & a & e & b & g & c \\ & & \uparrow & & & \\ v = & a & d & b & g & c \\ & & & & \uparrow & \\ w = & a & d & b & f & c \\ & & \downarrow & & & \\ u = & a & e & b & f & c \\ & & & & \downarrow & \\ q = & a & e & b & g & c \end{array}$$

Thus there needs to be an overlap of d and f. This overlap can either have one left hand side completely include the other — that is case a). Otherwise the sides overlap in the form abc with $ab = d$ and $bc = f$. This is case b).

If there is a prefix or suffix to abc in w then w is not minimal. □

COROLLARY 144: If local confluence fails at a minimal w, then $w = abcd$ such that b is not empty, either c or d is empty and abc and bd are left hand sides of rules.

The basic idea of the method now is that we inspect all such overlaps. If we have a failure of local confluence, i.e. we have $w \xrightarrow{*} u$ and $w \xrightarrow{*} v$ with u, v both reduced and $u \neq v$ we add a new rewriting rule $u \to v$ (or $v \to u$ if $u \prec v$).

This rule will not change the equivalence relation $\sim$, but it will remove this overlap problem.

When continuing on other overlaps, we need of course to consider also overlaps with this new rule. Thus this process may never terminate.

If it terminates (and one can show that it will terminate if there is a finite confluent rewriting system induced by $\mathcal{R}$), we have a confluent rewriting system, which allows us to calculate a normal form for the monoid presented by $\mathcal{R}$.

ALGORITHM 145: This method is called (after its proposers) the Knuth-Bendix[1] algorithm.

Input: A list L of rules of a rewriting system $\mathcal{R}$.

Output: L gets extended by deduced rules so that the rewriting system is confluent.

[1] Donald Knuth is also the author of "The Art of Computer Programming" and the creator of TEX. Peter Bendix was a graduate student.

Still in many cases when we want to change only a few points in an existing base this is too expensive. In such a situation it merits to modify an existing base. Let us assume that we know a base $B = (\beta_1, \ldots, \beta_m)$.

The easy case is if the new base is in fact a possible base image for G, i.e. the new base is $(\beta_1^g, \ldots, \beta_m^g)$ for $g \in G$. (Such an element g can be found easily, if it exists, using the stabilizer chain!)

In this situation, we can simply *conjugate* the whole stabilizer chain (i.e. conjugate all generators by g, take the image of all points under g) and obtain the desired chain.

In general (unless the group is the symmetric group), the new base $\Gamma = (\gamma_1, \ldots, \gamma_n)$ will not be a base image. In this situation we first try, using the base image approach, to move some base points in B to points in Γ, preferably at the same position, but even different positions are fine. Call this new base E.

Then we add the remaining points of Γ to E, by introducing trivial stabilizer steps (i.e. we have orbit 1 and all generators are Schreier generators). This is certainly possible on some level of the chain, but it might be the bottom level. The resulting base is called H.

Next we use a *base swap* procedure (see [HEO05, 4.4.7]) that will exchange the order of two subsequent base points η_i and η_j in H. (We only need to modify two subsequent entries in the stabilizer chain, as the previous and following stabilizers will be equal.)

Using this procedure, we move the base points in Γ (in the right order) to the start. Finally we delete trivial stabilizer steps at the end of the chain.

II.2. Consequences of Schreier-Sims

Using a stabilizer chain we can perform a variety of calculations for a group G:

- Test whether a permutation $g \in G$
- Given a base image $[\gamma_1, \ldots, \gamma_m]$ find, if possible, an element $g \in G$, such that $\beta_i^g = \gamma_i$: This is really just a modified element test in which we use the transversal elements corresponding to the base images.
- Calculate $|G| = \left|\beta_1^G\right| \cdot \left|G^{(1)}\right| = \left|\beta_1^G\right| \cdot \left|\beta_2^{G^{(1)}}\right| \left|G^{(2)}\right| = \cdots$ as the product or the orbit lengths.
- Normal Closure with proper element test.
- Determine the sizes of groups in the derived series $D_0 = G, D_i = D_{i-1}'$ and lower central series $L_0 = G, L_i = [G, L_{i-1}]$.
- Determine whether G is solvable or nilpotent.
- Test whether two elements are in the same coset of a subgroup.
- Determine the permutation action on the cosets of a subgroup.

- Determine the point wise stabilizer of a set (i.e. the subgroup stabilizing all points in the set) by calculating a stabilizer chain for a base starting with the points from the set.
- Enumerate G, i.e. assign to every element a number and have efficient functions to translate element to number and vice versa: We noted already that we can easily translate between elements and base images. We consider each base image as a list of numbers, according to the position of the point in the orbit. This is the "multi-adic" representation of a number $\in \{1, \ldots, |G|\}$.
- Obtain random elements with guaranteed equal distribution.

II.2.1. Factorization and Homomorphisms. We have noted before that the element test algorithm 23 will express a group element g as a product of transversal elements. On the other hand, every transversal element has been obtained as a product of the generators. By keeping track of how these transversal elements arose as products of the *original* generators, we can thus express any group element as a word in the generators.

NOTE 30: This looks like a perfect functionality for solving puzzles, such as Rubik's cube. Alas the words obtained are *horribly* long and in practice infeasible. One way used to obtain short words [Min98] is to add many short words in the original generators to the original generating set, thus automatically obtaining shorter words for stabilizer generators on lower levels.

A main use of this is in implementing homomorphisms. Suppose that G is a permutation group and we have a homomorphism $\varphi\colon G \to H$ given by a generating set $\underline{\mathbf{g}}$ of G and the images $\underline{\mathbf{g}}^{\varphi}$.

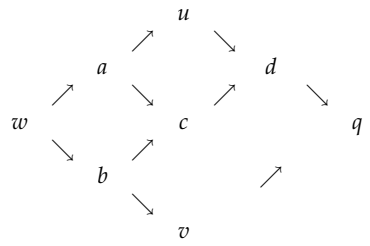
Then expressing an element $x \in G$ as word in $\underline{\mathbf{g}}$ lets us evaluate the same word in $\underline{\mathbf{g}}^{\varphi}$, which must be the image x^{φ}.

To speed up the way products are formed, we store also images for all Schreier generators – this way comparatively few products have to be evaluated. We obtain these images, by building a *new* stabilizer chain for G that is only used for the homomorphism. (As we can assume that $|G|$ is known, we can use a random Schreier-Sims algorithm with easy verification.)

The elements for which this chain are formed however are not elements of G, but elements of $G \times H$. We only consider the G-part for purposes of building the stabilizer chain, the H part just mirrors the multiplication.

The calculation then starts with a generating set of the form $\{(g, g^{\varphi}) \mid g \in \underline{\mathbf{g}}\}$.

Without loss of generality, we can assume that $u \neq w \neq v$ (otherwise we could set $q = u$ or $q = v$). Consider the first rewriting step of both deductions. We get $w \to a$ and $w \to b$ with $a \xrightarrow{*} u$ and $b \xrightarrow{*} v$.



Because we assume local confluence, we know that there is c with $a \xrightarrow{*} c$ and $b \xrightarrow{*} c$.

As w was chosen minimal in W, and $a \prec w$, we know that confluence cannot fail at a. Thus there is d such that $u \xrightarrow{*} d$ and $c \xrightarrow{*} d$. Therefore also $b \xrightarrow{*} d$.

By the same argument as before, confluence does not fail at b. Thus there is q such that $d \xrightarrow{*} q$ and $v \xrightarrow{*} q$. But then $u \xrightarrow{*} q$, which we wanted to show. □

IV.3. The Knuth-Bendix algorithm

As confluent rewriting systems solve the word problem, we would like to obtain such rewriting systems. In this section we will see a method that can be used to modify an existing rewriting system to become confluent. (If you know Gröbner bases, you will find this approach familiar.)

Theorem 142 tells us that for confluence we only need to ensure local confluence. Suppose that this does not hold, i.e. we have a word w with two reductions $w \to u$ and $w \to v$ but we cannot further rewrite both u and v to the same word q.

We want to consider "minimal" failure situations

LEMMA 143: Suppose that local confluence fails at w but not at any proper subword of w. Then one of the following holds:

a) w is the left hand side of a rule in $\mathcal{R}$ and contains the left hand side of another rule as subword (probably the whole of w).
b) $w = abc$ with nonempty a, b, c and ab and bc both are left hand sides of rules in $\mathcal{R}$.

COROLLARY 141: Let M be a monoid given by the rewriting system $\mathcal{R}$. If $\mathcal{R}$ has the Church-Rosser property, then the word problem in this monoid can be solved.

Proof: An element is trivial if its canonical representative is the empty word. □

Testing for the Church-Rosser property seems to be hard. However we will see now that it is in fact equivalent to local confluence, which is much easier to test.

THEOREM 142: For any rewriting system $\mathcal{R}$ with a reduction ordering the Church-Rosser property, confluence and local confluence are equivalent.

Proof: i)⇒ ii): Suppose that $\mathcal{R}$ has the Church-Rosser property and that $w, u, v \in F$ such that $w \xrightarrow{*} u$ and $w \xrightarrow{*} v$. Then $u \sim w \sim v$ and thus there exists q such that $u \xrightarrow{*} q$ and $v \xrightarrow{*} q$.

ii)⇒ i): Assume that $\mathcal{R}$ is confluent and that $u \sim v$. We want to find a q such that $u \xrightarrow{*} q$ and $v \xrightarrow{*} q$.

By definition 136 we have a sequence of words

$$u_0 = u, u_1, u_2, \ldots, u_{n-1}, u_n = v$$

such that $u_i \xrightarrow{*} u_{i+1}$ or $u_{i+1} \xrightarrow{*} u_i$.

We now proceed by induction on n. If $n = 0$ we can set $q = u = v$. If $n = 1$ we set q to be the smaller of u and v.

Thus assume $n \geq 2$. Then $u_1 \sim v$ and by induction there is a such that $u_1 \xrightarrow{*} a$ and $v \xrightarrow{*} a$. If $u_0 \xrightarrow{*} u_1$, we simply set $q = a$.

If instead $u_1 \xrightarrow{*} u_0$, by confluence there is q such that $u_0 \xrightarrow{*} q$ and $a \xrightarrow{*} q$. But then $v \xrightarrow{*} q$, as we wanted to show.

ii)⇒ iii): Obvious as local confluence is a special case of confluence.

iii)⇒ ii): Suppose that $\mathcal{R}$ is locally confluent but not confluent. Let W be the set of all words w, for which confluence fails. Because $\prec$ is a well-ordering, there is a smallest element $w \in W$.

Suppose that $w \xrightarrow{*} u$ and $w \xrightarrow{*} v$. We want to show that there is q such that $u \xrightarrow{*} q$ and $v \xrightarrow{*} q$, contradicting the failure of confluence.

Kernel: If H is also a permutation group, we can represent the direct product as a permutation group by moving the points on which H acts, i.e. for $S_3 \times S_4$ the element $((1,2),(3,4))$ is represented by $(1,2)(6,7)$. The domain Ω then decomposes in $\Omega_G \cup \Omega_H$.

Let $D = \left\langle (g, g^\varphi) \mid g \in \underline{\mathbf{g}} \right\rangle$ the group representing the homomorphism φ. Then the point wise stabilizer $\mathrm{Stab}_D(\Omega_H)$ corresponds to the set of elements whose image is trivial, i,e, its G-projection is the kernel of φ.

II.3. Backtrack

By "backtrack" we mean an algorithm that will run (in worst case) through all elements of a permutation group to find (one or all) elements fulfilling a certain property. Such an algorithm has exponential runtime in its input size, but is so far the best method known for tasks such as

- Centralizer and Normalizer in permutation groups
- Conjugating element in permutation groups
- Set stabilizer and set transporter
- Graph isomorphism

II.3.1. Basic backtrack. The basic version of backtrack simply runs through all group elements by using a stabilizer chain to enumerate all group elements in tree form: Each node corresponds to a (partial) base image. Its branches then are the images for the next base point, parameterized by the respective `.orbit` component. Figure 1 shows this enumeration for the example of $G = A_4$.

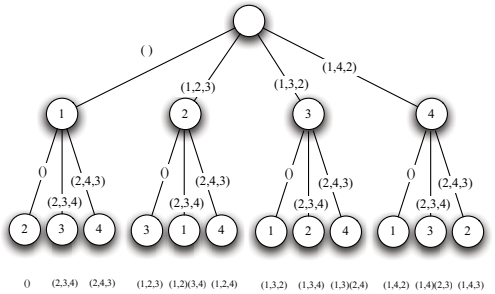
Again, as we consider stabilizer chains as recursive objects, this is a recursive algorithm.

Input: We are passing a (sub)chain (which describes the tree structure below) C and a partial product of representatives r, that describes the tree node.

Output: The program prints out all group elements

begin

1: $bot := |C.\texttt{stabilizer.generators}| = 0$;
2: $\Delta := C.\texttt{orbit}$;
3: **for** $\delta \in \Delta$ **do**
4: $\quad x := C.\texttt{transversal}[\delta]$;
5: $\quad$ **if** bot **then**
6: $\quad\quad$ Print $x \cdot r$;
7: $\quad$ **else**
8: $\quad\quad$ Call recursively for $C.\texttt{stabilizer}, x \cdot r$;
9: $\quad$ **fi**;
10: **od**;



Vertices are the images for the base point 1 and 2 respectively. Edge labels are the transversal elements. The permutations under the leafs are the resulting group elements.

FIGURE 1. Tree structure for A_4

end

We start with the whole chain for G and offset $r = ()$.

Obviously, instead of printing the element, we can test for whatever property we want.

In this version we are running always in the same way through the orbit. For several practical (see below) and aesthetic reasons, it can be desirable to run through elements in a lexicographic way (i.e. compare permutations as base images for the base $\{1, 2, 3, \ldots\}$). Then the possible images of the base point are given by the orbit points (that's what we chose) mapped under r (as we post-multiply by r).

We can achieve this by sorting Δ in line 2 according to the images under r: SortParallel($\{\delta^g \mid \delta \in \Delta\}, \Delta$).

II.3.2. Pruning. The problem of the basic backtrack routine is that it running through all elements of a group can be rather time intensive. A principal aim for any backtrack search is therefor to try to prune the search tree as aggressively as possible.

This possibility arises, because sometimes a partial base image already eliminates all elements which have these base point images.

An improved backtrack algorithm therefore will, every time a new base image is selected, employ a (problem-dependent) test, whether elements with this partial base image can in fact fulfill the desired property. Only if they can, lines 5-9 are executed.

ALGORITHM 138: Given a rewriting system $\mathcal{R}$, and a word u, find a reduced word v such that $u \xrightarrow{*} v$.

begin

```
1:  ok := true;
2:  while ok do
3:      ok := false;
4:      for Rules l → r in R do
5:          if l occurs in u then
6:              Replace u = alb by arb;
7:              ok := true;
8:          fi;
9:      od;
10: od;
```

end

Note that in general there are multiple ways to apply rules to a word. Thus in general reduced words are not automatically normal forms and $u \sim v$ does not imply that $u \xrightarrow{*} w$ and $v \xrightarrow{*} w$ for a unique element $w \in F$. Our aim is to rectify this.

IV.2. Confluence

We consider three slightly different properties which a rewriting system could have:

DEFINITION 139: A rewriting system $\mathcal{R}$ on F

i) has the *Church-Rosser property* if $u \sim v$ implies that there is $q \in F$ such that $u \xrightarrow{*} q$ and $v \xrightarrow{*} q$.

ii) is *confluent* if $w \xrightarrow{*} u$ and $w \xrightarrow{*} v$ imply that there is q such that $u \xrightarrow{*} q$ and $v \xrightarrow{*} q$.

iii) is *locally confluent* if $w \to u$ and $w \to v$ imply that there is q such that $u \xrightarrow{*} q$ and $v \xrightarrow{*} q$.

LEMMA 140: Suppose $\mathcal{R}$ has the Church-Rosser Property. Then every $\sim$ class contains a unique reduced element, the canonical representative for this class.

In particular, if we have $u \xrightarrow{*} v$ with v reduced, then v is uniquely determined by $\mathcal{R}$ and u.

Proof: Suppose that $u \sim v$ are both reduced. Then by the Church-Rosser property there exists q with $u \xrightarrow{*} q$ and $v \xrightarrow{*} q$. But as u and v are reduced we must have $u = q = v$. □

We now suppose that we have free monoid F consisting of words in the alphabet $\underline{\mathbf{f}}$.

We also assume to have a total ordering $\prec$ defined on F, which fulfills the following conditions:

i) $\prec$ is a well-ordering: Every nonempty set has a least element. This means that there are no infinite descending sequences.

ii) $\prec$ is translation invariant: If $a, b, c, d \in F$ and $a \prec b$ then $cad \prec cbd$. This implies that the empty word is the smallest element of F.

We call such an ordering a *reduction ordering*

EXAMPLE 133: Suppose that the elements of the alphabet $\underline{\mathbf{f}}$ are totally ordered. Then the "length-plus-lexicographic" ordering on F (i.e. first compare words by length and then lexicographically) is a reduction ordering.

DEFINITION 134: A *rewriting system* $\mathcal{R}$ on F is a collection of *rules* of the form $a \to b$ with $a, b \in F$ and $b \prec a$.

If we consider the rules of the rewriting system simply as relations, a rewriting system defines a monoid presentation. Similarly every monoid presentation yields a rewriting system in an obvious way. We will also talk about rewriting systems for groups, meaning the isomorphic monoid.

Given a rewriting system, we consider its rules as methods to "simplify" elements of F.

DEFINITION 135: If $u, v \in F$ we write $u \to v$ (with respect to $\mathcal{R}$) if there is a rule $a \to b$ in $\mathcal{R}$ such that u contains a as a substring (i.e. $u = xay$ with $x, y \in F$ and $v = xby$ is obtained by replacing a in u by b.

We write $u \xrightarrow{*} v$ if there is a sequence of words $u_0 = u, u_1, u_2, \ldots, u_{n-1}, u_n = v$ such that $u_i \to u_{i+1}$.

Because the ordering is translation invariant we have that $v \prec u$ in this case which justifies the idea of "simplification".

DEFINITION 136: We consider $u, v \in F$ to be equivalent with respect to $\mathcal{R}$, written $u \sim v$, if there is a sequence of words $u_0 = u, u_1, u_2, \ldots, u_{n-1}, u_n = v$ such that $u_i \xrightarrow{*} u_{i+1}$ or $u_{i+1} \xrightarrow{*} u_i$.

It is not hard to see that $\sim$ is in fact the finest equivalence on F defined by $\mathcal{R}$, thus the equivalence classes correspond to the elements of the finitely presented monoid defined by $\mathcal{R}$.

DEFINITION 137: A word $v \in F$ is called *reduced* if there is no $w \in F$ such that $v \to w$ (with respect to $\mathcal{R}$).

We need the fact that $\prec$ is a well-ordering to ensure that for u we can compute a reduced element v with $u \xrightarrow{*} v$ in finitely many steps.

EXAMPLE 31: As an example of such a test, suppose we are looking for an element that maps $(1,2)(3,4,5)$ to $(2,4)(1,5,3)$. We chose a base starting with $\{1,2\}$.

As an n-cycle must be mapped to an n-cycle, the image of 1 can be only 2 or 4, eliminating all top branches but two. Furthermore, if $1^g = 2$, we know that $2^g = 4$; respectively $1^g = 4$ implies $2^g = 2$. On the second level we thus have but one branch.

Similar restrictions will hold for the next base points.

EXAMPLE 32: We want to find the centralizer of $(1,2,4)(5,6,8)$ in $G = \langle (1,3,5,7)(2,4,6,8), (1,3,8)(4,5,7) \rangle$. This group has order 24, we pick base $(1,2)$ and get the chain:

```
rec( generators := [ (1,3,5,7)(2,4,6,8), (1,3,8)(4,5,7) ],
  orbit := [ 1, 3, 5, 8, 7, 2, 4, 6 ],
  transversal := [ (), (1,2,7,5,6,3)(4,8), (1,3,5,7)(2,4,6,8),
      (1,4,2)(5,8,6), (1,5)(2,6)(3,7)(4,8), (1,6,7)(2,3,5),
      (1,7,5,3)(2,8,6,4), (1,8,2,5,4,6)(3,7) ],
  stabilizer := rec( generators := [ (2,8,7)(3,6,4) ],
      orbit := [ 2, 8, 7 ],
      transversal := [ , (),,,,, (2,7,8)(3,4,6), (2,8,7)(3,6,4) ],
      stabilizer := rec( generators := [  ] ) ) )
```

We can map 1 to $1, 2, 4, 5, 6, 8$. In each case the image of 2 is then fully determined:

1^g	2^g	x	Works?
1	2	()	$\surd$
2	4	(1,2,4)(5,6,8)	$\surd$
4	1	(1,4,2)(5,8,6)	$\surd$
5	6	(1,5)(2,6)(3,7)(4,8)	$\surd$
6	8	(1,6,4,5,2,8)(3,7)	$\surd$
8	5	(1,8,2,5,4,6)(3,7)	$\surd$

At this point we have actually found *all* elements in the centralizer.

EXAMPLE 33: If we want to find the *set-wise* stabilizer of $\Delta \subset \Omega$ we choose a base whose points are chosen within Δ as far as possible. Suppose that $\beta_1, \ldots, \beta_k \in \Delta$, and $G^{(k)}$ moves no point in Δ. Thus clearly $G^{(k)} \leq \mathrm{Stab}_G(\Delta)$ and the possible images of β_i ($i \leq k$) are restricted to Δ.

Such pruning conditions obviously are problem specific. When intelligently applied, however they can often eliminate large parts of the search space. This usually also requires a suitable choice of base.

In the example of finding conjugating permutations, one would try to chose the first base point within a cycle whose length occurs rarely and

then choose subsequent base points from the same cycle, as after choosing the first image the subsequent images all are uniquely determined.

In the following discussion we will assume that we have chosen a suitable base, and that we are doing such problem-specific pruning.

NOTE 34: Newer version of backtrack algorithms, so called "Partition backtrack" routines label the tree not with base images, but with partitions of Ω (with ordering relevant): The partial base image $(\gamma_1, \ldots, \gamma_k)$ then corresponds to a partition with Each γ_i $(i \leq k)$ is in its own cell.

Properties of the problem however can give a further splitting of the remaining cell which do not affect the current base point. For example when centralizing $(1,2,3)(4,5,6)$ if we stabilize 1 we also have to stabilize 4.

We can describe such conditions by intersection with partitions.

The effect of this is that the tree of the backtrack search becomes more shallow.

II.3.3. Properties defining subgroups. For most properties interesting in a group-theoretic context, the set of elements fulfilling the condition we search for actually forms a subgroup, respectively a double coset. (A *double coset* is a subset of elements of the form $SgT = \{sgt \mid s \in S, t \in T\}$ for $S, T \leq G$.) For example:

- Centralizer, Normalizer, set stabilizer, automorphism group of a graph are subgroups.
- In a conjugacy test: Find g with $a^g = b$ the fulfilling elements are in a double coset $C_G(a) \cdot h \cdot C_G(b)$ if h is one solution.
- Testing for isomorphism between the graphs Γ and Θ, if h is one isomorphism, the set of isomorphisms has the form $\mathrm{Aut}(\Gamma)h\,\mathrm{Aut}(\Theta)$.

In the following we will consider only the situation of a subgroup (the double coset situation is similar): We are looking to find all elements in G that fulfill a testable property. We assume that this set of elements forms a subgroup $P \leq G$.

In fact however we only need to find a generating set of P (and chances are good that a few elements of P we found generate P), then in fact much time will be spent in proving that no element*outside* the subgroup we found so far fulfills the property. So let us suppose we found a subgroup $K \leq P$ (which might be trivial). Also whenever we find a new element $g \in P$, we update $K := \langle K, g \rangle$.

NOTE 35: When testing for a single "mapping" element (e.g. in a conjugacy test) of course we are not deliberately building such a subgroup K. However we can still do so if we happen to come upon an element stabilizing the initial object. This way similar benefits are obtained.

CHAPTER IV

Rewriting

Rewriting is the formal context in which we use a presentation to bring elements into normal form. To make this deterministic we will consider relations instead of relators and consider them as *rules* from→to.

Much of the material in this chapter is from [Sim94]

If you know the basic theory behind Gröbner bases, much of this will look familiar.

IV.1. Monoids and Rewriting Systems

DEFINITION 130: A *monoid* is a set with an associative binary operation and an identity element. (In other words: we drop the condition on inverses.)

If $\{x_1, \ldots, x_n\}$ is an alphabet, we consider the set of words (including the empty word) over this alphabet. With concatenation as operation they form a monoid.

We define finitely presented monoids analogous to finitely presented groups. Note however that due to the lack of inverses we have to write in general relations instead of relators.

LEMMA 131: Every group is a monoid and every finitely presented group is a finitely presented monoid.

Proof: Finitely presented is the only thing that needs showing: If $G = \left\langle \underline{\mathbf{g}} \mid R \right\rangle$ we form a monoid generating set $\underline{\mathbf{m}} = \{g_1, g_1^{-1}, \ldots\}$ (with g_i^{-1} understood as a formal symbol). Then a monoid presentation is

$$\left\langle \underline{\mathbf{m}} \mid \{r = 1 \mid r \in R\} \cup \{g_i g_i^{-1} = 1, g_i^{-1} g_i = 1 \mid 1 \leq i \leq m\} \right\rangle$$

□

PERFORMANCE 132: For finite groups one can often do better, as relations of the form $a^m = 1$ imply the existence of inverses.

As our strategy is "left-first", i.e. we first enter the stabilizer of a base point, before considering any cosets, we will have examined the whole of $G^{(i)}$ before considering any other elements of $G^{(i-1)}$. In particular, we can assume that we know $G^{(i)} \cap P$ before testing any element of G outside $G^{(i)}$.

NOTE 36: This observation also shows that the backtrack search will automatically produce a strong generating set for P (or the subgroup K of elements found so far). We can thus assume at little cost that we have a stabilizer chain for K (and in the end for P).

LEMMA 37: Suppose we know $K = G^{(l)} \cap P$ and that $\mathcal{N}$ is a node which prescribes the first l base images. Suppose we find an element g below $\mathcal{N}$ that is in P. Then we can discard the whole remaining subtree below $\mathcal{N}$.

Proof: Any further element of P in this subtree is in the coset Kg. □

This test works if we find new elements, but we can do much better: Suppose we test an element $g \notin K$. Then either $g \in P$, in which case we increase K by at least a factor 2. Or $g \notin P$, but then no element in the double coset KgK can be in P either.

While this condition has the potential to reduce the search space enormously (making the cost more proportional to $\left[G{:}P\right]$ than to $|G|$, the problem is just how to incorporate it in the backtrack search.

What we want to do is to test every double coset KgK only once. A standard method for such duplicate rejection (without explicitly storing all elements of KgK for every g tested) is to define a "canonical" representative for each double coset. Then every element g that is not canonical for its double coset can be discarded (as we will test the – different – canonical representative at another time).

Typically the definition of "canonical" will require some arbitrary symmetry-breaking condition (all elements are images under a group, so they are in some way "the same"). What we will use is that the element is minimal with respect to a comparison of base images (i.e. we lexicographically compare the base images $(\beta_1^g, \beta_2^g, \ldots)$) among all elements in the double coset. Note that by sorting the orbits the basic backtrack algorithm will run through elements in this ordering.

Unfortunately finding the smallest element in a double coset (or testing whether one element is smallest) is hard. We will thus use weaker conditions, that adapt well to the tree traversal strategy.

The first condition does in fact only use minimality in the left coset gK:

LEMMA 38: Suppose that $\mathcal{N}$ is a node in the search tree that prescribes the first l base images as $(\gamma_1, \ldots, \gamma_l)$ and that $K \leq P$ is the subgroup found so

far. If g lies under $\mathcal{N}$ and is the smallest element of KgK then γ_l is minimal in the orbit $\gamma_l^{\mathrm{Stab}_K(\gamma_1,\ldots,\gamma_{l-1})}$.

Proof: Suppose not. Let $h \in \mathrm{Stab}_K(\gamma_1,\ldots,\gamma_{l-1})$ such that $\gamma_l^h < \gamma_l$. Then $gh \in KgK$ and $gh < g$, contradiction. □

To use this lemma we need to perform a base change to find the stabilizer $Stab_K(\gamma_1,\ldots,\gamma_{l-1})$. Note that we will already know $Stab_K(\gamma_1,\ldots,\gamma_{l-2})$, so little extra work is needed.

The next criterion eliminates certain base images to test.

LEMMA 39: Suppose that $\mathcal{N}$ is a node in the search tree that prescribes the first l base images as $(\gamma_1,\ldots,\gamma_l)$ and that $K \leq P$ is the subgroup found so far. Let $R := Stab_G(\gamma,\ldots,\gamma_{l-1})$, $S := Stab_K(\beta_1,\ldots,\beta_{l-1})$, and $s = \left|\beta_l^S\right|$.

If g lies under $\mathcal{N}$ and is the smallest element of KgK then γ_l cannot be among the last $s-1$ elements of its orbit under R.

Proof: Let $\Gamma = \{\beta_l^{hg} \mid h \in S\} = (\beta_l^S)^g$. Then $|\Gamma| = s$ and $\gamma_l = \beta_l^g \in \Gamma$.

As any product $hg \in Kg \subset KgK$, the minimality of g implies that $\gamma_l = \min\Gamma$.

If $\gamma = \beta_l^{hg} \in \Gamma$, then $\gamma^{g^{-1}h^{-1}g} = \gamma_l$ with $g^{-1}h^{-1}g \in R = (G^{(l-1)})^g$. Thus $\Gamma \subset \gamma_l^R$ and γ_l^R must contain at least $s-1$ elements larger than γ_l. □

II.4. Natural Actions and Decompositions

If we have a permutation group, we often get "induced" actions that will in lead to particular decompositions of the group. These decompositions are of independent interest, but they are most conveniently presented here.

We will be talking about permutation groups G. If G is a group with a *permutation action* φ on Ω, all statements remain true if we interpret them for the factor $G/\operatorname{Kern}\varphi$.

II.4.1. Orbits: Intransitive Groups. The first situation we want to look at is that of an intransitive group, i.e. a permutation group that has multiple orbits on its permutation domain:

Suppose we have that $\Omega = \Delta \cup \Gamma$ and both Γ and Δ are (unions of) orbits. In this situation be get two homomorphisms, $\alpha\colon G \to S_\Gamma$ and $\beta\colon G \to S_\Delta$, such that $\operatorname{Kern}\alpha \cap \operatorname{Kern}\beta = \langle 1\rangle$.

We set $A = G^\alpha$ and $B = G^\beta$

We now form a new homomorphism, $\epsilon\colon G \to A \times B$, defined by $g^\epsilon = (g^\alpha, g^\beta)$. Then $\operatorname{Kern}\epsilon = \operatorname{Kern}\alpha \cap \operatorname{Kern}\beta = \langle 1\rangle$.

Such an algorithm is sometimes called a "Las Vegas" algorithm (in analogy to "Monte Carlo" algorithms): We have a randomized computation of a result that may be wrong, but can do a subsequent verification. (The runtime of such an algorithm thus is good in average, but can be unbounded in the worst case of repeated verification failure.)

The basic approach is the following (again much of the later steps is not implemented):

1. Compute a randomized stabilizer chain for G.
2. Using this chain compute a composition series. (As part of this we get for each factor $G_i > G_{i+1}$ in this series an epimorphism $G_i \to G_i/G_{i+1}$.)
3. Using constructive recognition of the simple factors (see VI.3), write down a presentation for each simple factor F.
4. Use the method of lemma 127, construct a presentation for G. If the initial chain was too small this is in fact a presentation for a smaller group.
5. Verify that the elements of G actually fulfill the presentation.

To obtain a good runtime complexity for permutation group algorithms in general, we want these steps all to be "fast" (in terms of the degree of the initial permutation group G). This means in particular: We need to be able to construct isomorphisms for the simple factors "quickly" (which in fact has been proven) and need to obtain "short" presentations for the simple factors (basically of relator length $\log^2|F|$).

The Steinberg presentations mentioned in the last section do not fulfill this, but for almost all cases short variants are known [BGK+97, HS01]. Only the so-called Ree-groups (Lie Type ${}^2G_2$) are missing so far.

III.10.3. The simple case. While we could easily write down a presentation for a cyclic factor, in general one will still need presentations for the simple composition factors.

One way (which is currently used in GAP) is to use the method of section III.10.1. For small composition factors this produces reasonable presentations (albeit nothing to boast about).

A much better approach is — mirroring how one would prove theorems — to use the vast amount of theoretical information that has been obtained (for example in the course of the classification of finite simple groups) about simple groups.

If we go through the classes of nonabelian finite simple groups, the following information is found in the literature:

Alternating Group: It is a not too hard exercise to show that for odd n, A_n is generated by the elements $g_1 = (1,2,n)$, $g_2 = (1,3,n)$, $\ldots$, $g_{n-2} = (1,n-1,n)$ and that

$$\left\langle g_1, \ldots, g_{n-2} \mid \forall i,j > i : g_i^3 = (g_i g_j)^2 = 1 \right\rangle$$

is a presentation.

Groups of Lie Type: This class includes the groups coming from matrix groups, such as $PSL_n(q)$. The unified way to construct these groups also offers a "generic" way to write down a presentation ("Steinberg-presentation").

Sporadic Groups: Finally there are 26 so-called "sporadic" groups that do not fit in the previous classes (they include for example the Mathieu groups). For these ad-hoc presentations are known.

An excellent source for such information is the ATLAS of simple groups [CCN+85].

Given a simple composition factor A, we construct an isomorphism (for example by a variant of algorithm 102) to an isomorphic group B in "nice" form, for which we can just write down the presentation. This lets us transfer the presentation to A.

We will see more efficient ways of constructing such isomorphisms later in section VI.3.

Alas very little of this approach is actually implemented.

III.10.4. Upgrading Permutation group algorithms to Las Vegas. We have seen before that fast algorithms for permutation groups rely on randomized computation of a stabilizer chain and therefore may return a wrong result. To rectify this one would like to have a subsequent step that will verify that the chain is correct. If now, we then can simply continue with further random elements until a renewed test verifies correctly.

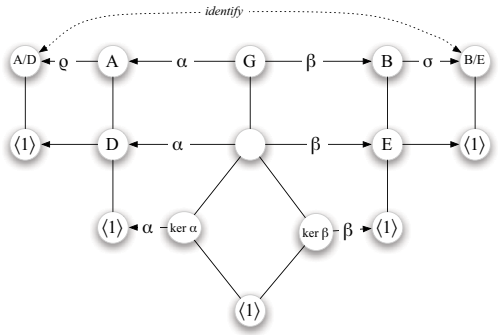


FIGURE 2. Subdirect Product

We can thus consider G as (isomorphic to) a subgroup of $A \times B$. Such a group is called a *subdirect product*, the construction is due to REMAK [Rem30].

(We do not really need that G is a permutation group, we just have two homomorphisms, whose kernels intersect trivially; respectively two normal subgroups which intersect trivially.)

We now want to make this construction synthetic, i.e. we want to describe Image(ϵ) in terms of A and B.

For this we set $D = (\mathrm{Kern}\,\beta)^\alpha \lhd A$ and $E = (\mathrm{Kern}\,\alpha)^\beta \lhd B$. Then $A/D \cong G/\langle \mathrm{Kern}\,\alpha, \mathrm{Kern}\,\beta\rangle \cong B/E$, i.e. we have isomorphic factor groups of A and B. See figure 2 for detail.

Let $\rho\colon A \to A/D$ and $\sigma\colon B \to B/E$ the natural homomorphisms and $\zeta\colon A/D \to B/E$ the isomorphism given by $(g^\alpha)^\rho \mapsto (g^\beta)^\sigma$. Then we get that

$$G^\epsilon = \left\{ (a,b) \in A \times B \mid (a^\rho)^\zeta = b^\sigma \right\}$$

This construction will work for any groups A and B, which have isomorphic factor groups. If $A/D \cong B/E$ given by an isomorphism ζ, we can form the set

$$A \perp B = \left\{ (a,b) \in A \times B \mid (a^\rho)^\zeta = b^\beta \right\} \leq A \times B$$

It is an easy exercise to see that this set $A \perp B$ is a group.

Note that the notation is misleading, the product depends on the choice of factor groups and on ζ. We say that it is the subdirect product in which the factor groups A/D and B/E are "glued together".

Returning to the situation of a permutation group, we see that every group with two orbits is a subdirect product of two transitive groups. In

general every permutation group is obtained by forming iteratively subdirect products of transitive groups.

To classify all permutation groups of a given degree we thus would need to:

- Classify all transitive groups up to this degree.
- Form their subdirect products

II.4.2. Blocks: Imprimitive Groups. As we can consider intransitive groups as subdirect products of transitive groups, let us now consider a group G acting transitively on Ω.

DEFINITION 40: A partition $\mathcal{B} = \{B_1, \ldots, B_k\}$ of Ω is a *block system*, if it is invariant under G. I.e. the set-wise image $B_i^g \in \mathcal{B}$ for every $g \in G$.

NOTE 41: The following two block systems always exist. They are called the *trivial block systems*:

$$\mathcal{B}_1 = \{\{\omega\} \mid \omega \in \Omega\}, \qquad \mathcal{B}_\infty = \{\Omega\}$$

DEFINITION 42: A group is acting *imprimitively* on Ω if G acts transitively, and affords a nontrivial block system. Otherwise we say the group acts *primitively*.

LEMMA 43: Let $\mathcal{B} = \{B_1, \ldots, B_k\}$. Then for every i, j there exists $g \in G$, such that $B_i^g = B_j$. In particular $|B_i| = |B_j|$ and thus $|\Omega| = |B_1| \cdot |\mathcal{B}|$.

Proof: Let $\delta \in B_i$ and $\gamma \in B_j$. As G acts transitively, there is $g \in G$ such that $\delta^g = \gamma$. Thus $B_i^g \cap B_j \neq \emptyset$. As the partition is kept invariant we have that $B_i^g = B_j$. □

COROLLARY 44: A block system is determined by one block – the other blocks are just images.

COROLLARY 45: Any transitive group of prime degree is primitive.

LEMMA 46: Suppose G acts transitively on Ω and let $S = \mathrm{Stab}_G(\omega)$ for some $\omega \in \Omega$. Then there is a bijection between subgroups $S \leq T \leq G$ and block systems $\mathcal{B} = \{B_1, \ldots, B_k\}$ for G on Ω.

Using the convention that B_1 is the block containing ω, the bijection s given by $T = \mathrm{Stab}_G(B_1)$, $B_1 = \omega^T$.

Proof: Suppose that $S \leq T \leq G$. We set $B = \omega^T$ and $\mathcal{B} = B^G$ and claim that $\mathcal{B}$ is a block system:

Let $g, h \in G$, and suppose that $B^g \cap B^h \neq \emptyset$. Then there exists $\delta, \gamma \in B$ such that $\delta^g = \gamma^h$. As $B = \omega^T$ we have that $\delta = \omega^s$, $\gamma = \omega^t$ for $s, t \in T$. Thus $\omega^{sg} = \omega^{th}$, and thus $sgh^{-1}t^{-1} \in \mathrm{Stab}_G(\omega) = S \leq T$. This implies that

Proof: It is easily seen that the relations all hold in G. To show that the presentation does not define a larger group, observe that the relations in R_4 ($h_j^{-}1m_ih_j =$ word in $\underline{\mathbf{m}}$ implies $m_ih_j = h_j \cdot$ word in $\underline{\mathbf{m}}$) permit us to write every element in the presented group as a word in $\underline{\mathbf{h}}$ with a word in $\underline{\mathbf{m}}$. The relations in R_3 show that (up to changes in $\underline{\mathbf{m}}$) every word in $\underline{\mathbf{h}}$ can be transformed to one of $|G/N|$ possibilities. The relations in R_1 similarly reduce the words in $\underline{\mathbf{m}}$ to $|N|$ classes. Thus the presentation defines a group of order $\leq |G|$. □

Using this lemma and a composition series of G, we can form a presentation for G based on presentations of the composition factors (see the next section for these).

PERFORMANCE 128: In practice one often gets a nicer presentation and faster performance by using a chief series of G and using the (obvious) presentations for direct products of simple groups.

Pc presentations. It is worth noticing a special case of this which occurs when G is solvable. Then the composition series consists of cyclic factors of prime order and we trivially get a presentation $\langle g | g_p = 1 \rangle$ for these cyclic factors. We therefore get a presentation of the following form:

- Assuming the composition series is $G = G_0 > G_1 > \ldots > G_n = \langle 1 \rangle$, we have generators $g_1, \ldots, g_n$ with $G_{i-1} = \langle G_i, g_i \rangle$.
- We get *power relations* for R_3: If $\left[G_{i-1}{:}G_i\right] = p_i$, we have that $g_i^{p_i}$ can be expressed as a word in the generators g_{i+1} and following
- For R_4 we get *conjugacy relations*: If $i < j$ we have that $g_j^{g_i}$ can be expressed as a word in g_{i+1} and following. (In fact one can do better if the group is supersolvable or nilpotent.)

DEFINITION 129: Such a presentation is called a *PC-presentation* (with "PC" standing alternatively for "polycyclic", "power-conjugate" or "power-commutator".)

We observe that the relations in R_4 permit us to order generators, the power relations in R_3 restrict exponents. Thus every element of G can be written (uniquely) as a product $g_1^{e_1} g_2^{e_2} \cdot \cdots \cdot g_n^{e_n}$ with $0 \leq e_i < p_i$.

We can thus represent group elements by an *exponent vector* $[e_1, \ldots, e_n]$ which is a very compact form of storage. Section IV.3.1 will describe how one can perform arithmetic operations on such exponent vectors.

In GAP one can convert a (solvable) permutation group into such a form using the command `IsomorphismPcGroup`.

it as a word in the generators of S. The corresponding relator would have enforced equality of x and y and thus is added to the set of relators.

By the same argument as before, the result will be a presentation for G. We can use Tietze-transformations to eliminate the generators of S and obtain a presentation purely in the generators of G though typically of longer total relator length.

We can iterate this process over a chain of subgroups. In particular we can do this for the subgroups in a stabilizer chain and get a presentation in a strong generating set.

An application of this is a test whether a map from a permutation group to another group, given by generator images, extends in fact to a homomorphism. Construct a stabilizer chain for the prospective homomorphism. then proceed as if constructing a presentation. Instead of adding relators, check whether the relators evaluate trivially in the generator images.

NOTE 126: We can use this method as well, if we want to verify a stabilizer chain that has been obtained with random methods, and might indicate the group being too small: Using this chain, we compute a presentation and then check that the group generators fulfill this presentation. If the chain was too small they will not. This yields the so-called "Todd-Coxeter-Schreier-Sims" algorithm mentioned in section II.1.7.

Despite the fact that the Todd-Coxeter method has no bounds on the runtime whatsoever, this produces a respectable performance in practice. (See also section III.10.4.

III.10.2. Using the extension structure. The presentations obtained with this method often are rather messy. It therefore often makes sense to use more information about the composition structure of G and to build a presentation for G from presentations for its composition factors.

The cost of this is that we get a presentation in a new generating set. In most applications this is of little concern.

The heart of this method is the following easy lemma:

LEMMA 127: Let $N = \langle \underline{\mathbf{n}} \rangle \lhd G$ and $G = \left\langle N, \underline{\mathbf{g}} \right\rangle$. Suppose that $N = \langle \underline{\mathbf{m}} \mid R_1 \rangle$ is a presentation for N and that $G/N = \langle \underline{\mathbf{h}} \mid R_2 \rangle$ is a presentation for G/N such that $h_i = Ng_i$. For an element $x \in N$ let $\varrho(x)$ be the expression of x as a word in $\underline{\mathbf{m}}$.

Then the following is a presentation for G:

$$\langle \underline{\mathbf{h}} \cup \underline{\mathbf{m}} \mid R_1 \cup R_3 \cup R_4 \rangle$$

where $R_3 = \{r(\underline{\mathbf{h}})/\varrho(r(\underline{\mathbf{g}})) \mid r \in R_2\}$ and $R_4 = \{m_i^{h_j}/\varrho(n_i^{g_j}) \mid i, j\}$.

$gh^{-1} \in T$. As T stabilizes B (by definition), we thus have that $B^g = B^h$. Thus the images of B under G from a partition of Ω. As orbit, this partition is clearly G-invariant.

Vice versa, let $\mathcal{B}$ be a block system and let $\omega \in B \in \mathcal{B}$ be the block containing ω. Then any $g \in G$ which fixes ω has to fix B, thus $\mathrm{Stab}_G(\omega) \leq \mathrm{Stab}_G(B)$.

Now, observe that if B is a block and $\delta, \gamma \in B$, there is $g \in G$ such that $\delta^g = \gamma$. But then $\gamma \in B^g$, thus $B = B^g$ and $g \in \mathrm{Stab}_G(B)$. Thus $\omega^{\mathrm{Stab}_G(B)} = B$.

Vice versa, if $S \leq T \leq G$ and $B = \omega^T$ and $x \in \mathrm{Stab}_G(B)$ then $\omega^x = \omega^t$ for $t \in T$. Thus $xt^{-1} \in S \leq T$ and thus $x \in T$ which shows that $\mathrm{Stab}_G(\omega^T) = T$ which shows that we have a proper bijection. □

COROLLARY 47: A transitive permutation group is primitive if and only if a point stabilizer is a maximal subgroup.

Respectively: A subgroup $S \leq G$ is maximal if and only if the action on the cosets of S is primitive.

II.4.3. Finding Blocks. The following algorithm to find block systems is due to [Atk75]. Its core is a method that for a given $\alpha \in \Omega$ determines the finest block system in which 1 and α are in one block. By running through all possible α, we thus find the minimal blocks. This is done using the following easy rule:

LEMMA 48: If B is block in a block system for G, and $\alpha, \beta \in B$ and $g \in G$ then α^g, β^g are in the same block.

NOTE 49: As blocks correspond to subgroups containing the point stabilizer it is easy to build all blocks from these: If $1 \in B_1$ and $1 \in B_2$ are blocks in two block systems, we use the same algorithm to find the finest block system, in which $B_1 \cup B_2$ is a subset of one block and so on.

ALGORITHM 50: This algorithm finds the finest block system, in which a block fully contains *seed* $\subset \Omega$. We call it with *seed* $= \{1, \alpha\}$ to obtain minimal blocks. We identify blocks, by storing for each point $\omega \in \Omega$ one element of the block containing ω as block representative $r[\omega]$. We will use these representatives to identify (partial) blocks. A subroutine is used to form the union of such partial blocks:

UNION(α, β)

Input: Two partial blocks, given by their representatives α and β.
Output: The two blocks are joined. The identifier of the partial block that
begin
1: for $\omega \in \Omega$ do
2: if $r[\omega] = \beta$ then

```
 3:       r[ω] := α;
 4:    fi;
 5: od;
end
```

NOTE 51: The merging of classes is a standard problem in computer science ("Union-find") and (more) efficient algorithms for this are discussed in textbooks.

With this we get the actual block system finding algorithm:

Input: A group $G = \left\langle \underline{\mathbf{g}} \right\rangle$ acting transitively on Ω. A subset $seed \subset \Omega$.

Output: The finest block system in which all points of *seed* are together in one block.

```
begin
 1: r := [];
 2: q := []; {A queue of points that have changed their block}
 3: μ := seed[1]
 4: for ω ∈ Ω do
 5:    if ω ∈ seed then
 6:       r[ω] := μ;
 7:       Add ω to q;
 8:    else
 9:       r[ω] := ω;
10:    fi;
11: od;
12: l := 1;
13: while l ≤ |q| do
14:    γ := q[l]; δ := r[γ]; {point and its representative}
15:    for g ∈ g do
16:       α := r[γ^g];
17:       β := r[δ^g];
18:       if α ≠ β then {Two points are in the same block but their images are
          not}
19:          UNION(α, β); {join block given by β to block given by α}
20:          Add β to q; {As β block got deleted}
21:       fi;
22:       l := l + 1;
23:    od;
24: od;
25: return r;
end
```


III.9.1. Abelianized rewriting. We can combine the algorithms of this section and the previous one and ask for the abelian invariants of a subgroup. Instead of performing one algorithm after the other, rewriting relators and then abelianizing them, it is beneficial to immediately abelianize relators while rewriting. This avoids maintaining long relators intermediately and leads to a much more nimble performance.

Determining the abelianization of a subgroup is one of a handful methods know for determining the infinity of certain groups (there is no universal method): Find a subgroup (of smallish index) whose abelian quotient is infinite.

III.10. Getting a Presentation for a permutation group

In some situations we have a group G already given as a permutation group, but want to obtain a presentation for G. This occurs for example when computing complements to a normal subgroup (see IV.4.1) or to test whether a map on generators extends to a homomorphism.

In GAP such functionality is provided by the following commands: `IsomorphismFpGroup` lets GAP choose the generating system in which the presentation is written (typically yielding more generators but a nicer presentation). `IsomorphismFpGroupByGenerators` produces a presentation in a particular generating system.

III.10.1. Reverse Todd-Coxeter. A basic algorithm is due to [Can73], it might be considered easiest as a reversal of coset enumeration:

Suppose we start a coset enumeration for G acting on the cosets of the trivial subgroup, starting without relators. We write t_x to denote the representative for coset x as given by the coset table.

At some point we will define a new coset y which is in fact equal to an already existing coset x. Thus $t_x t_y^{-1}$ must be trivial in G and thus must be a relator. We add this as a relator to the enumeration process (and fill in the corresponding relator table as far as possible). We continue with this until we get and up with a complete table.

Clearly we only added valid relators for G. On the other hand these relators define a group which (as we can see by performing a coset enumeration by the trivial subgroup) has the same order as G, thus the relators yield a presentation for G.

In a variation, suppose that $S \leq G$ and that we know already a presentation of S. We now form a presentation on the generators for S together with the generators for G. As relators we start with the known relators for S as well as relators that express the generators for S as words in the generators for G. Then start a coset enumeration for the cosets of S in G. If two cosets x and y seem different we know that $t_x t_y^{-1} \in S$, thus we can express

The principal idea is the following observation: Suppose that F is the free group in which the presentation for G is given and $\varphi\colon F \to G$ is the epimorphism. Let $N = F' = \langle x^{-1}y^{-1}xy \mid x, y \in F\rangle_F$. then $N^\varphi = G'$. Thus $F/N \cdot \operatorname{Kern}\varphi \cong G/G'$.

We thus get a description for G/G' by simply *abelianizing* the presentation for G, i.e. considering it as a presentation for an abelian group.

As the generators in an abelian group commute, we can describe the relators for G/G' by a matrix A: The columns correspond to the generators of G/G' (images of the generators of G), each row represents one relator, which we can assume to be in the form $g_1^{e_1}g_2^{e_2}\cdots g_m^{e_m}$, we simply store the exponent vector $[e_1, \ldots, e_m]$.

Now consider the effect of elementary transformations over $\mathbb{Z}$ on the rows and columns of A (i.e. swap, adding a multiple of one to another and multiplication by ± 1): Such transformations on the rows correspond to a change of the relator set R, but (as they are invertible) these new relators will generate the same group. Transformations of the columns correspond to a generator change for G/G'. Again the invertibility of the transformation shows that the new elements still generate the whole of G/G'.

We now recall, that we can use such transformations to compute the *Smith Normal Form* of A, i.e. we can transform A into a diagonal matrix S (possibly even with divisibility conditions among the diagonal entries).

This new matrix S will describe a group isomorphic to G/G'. As S is diagonal, this group is simply a direct product of cyclic groups of orders given by the diagonal entries (using order ∞ for entry 0).

If we do not only compute the Smith Normal form S of A, but also determine matrices $A = P \cdot S \cdot Q$, the matrix Q describes the necessary change of the generating system, thus Q^{-1} describes how to form a homomorphism $G \to C$ with $C \cong G/G'$ the abelian group given by the diagonal entries in S.

PERFORMANCE 125: The bottleneck of such a calculation is that — even if the entries in S are small — the calculation of the Smith Normal Form can often produce intermediate coefficient explosion. (It becomes even worse for the (non-unique!) transformation matrices P and Q.) There is an extensive literature considering strategies for such calculations (in particular on how to keep entries in P and Q small).

To indicate the difficulty, note that the standard approach of reduction modulo a prime does not work, because we can always scale modulo a prime. One way to rescue this is to use a theorem relating the diagonal entries of S to the gcd's of determinants of minors of A, and calculating these determinants modulo a prime. This however does not yield transforming matrices.

Proof: Clearly the partition given by r can never be coarser than the minimal block system given by *seed* as we only join classes that must be in the same block. We thus need to show only that the partition returned is invariant under G, i.e. we have to show that if $\omega, \delta \in \Omega$ are in the same block and $g \in \mathbf{g}$, then ω^g and δ^g are in the same block.

We ensure this by enforcing the following condition:

> (*) If β is the label for a (partial) block, and ω is in this block, then β^g and ω^g are in the same block.

Clearly it is sufficient to enforce enforce condition (*) for all points which changed the label of their (partial) block.

Thus, if in line 20 we would add *all* points of the partial block labeled by β to the queue, condition (*) is enforced by the `while` loop in line 13-24.

Finally consider the case of a point ω that is labeled by β: Suppose we relabel ω to α. This can only happen if we relabel β to α and in this case we enforce (*) for β and α.

But as ω got relabeled at the same time, and as we already enforced (*) for ω and β, this will automatically enforce (*) for ω and α.

It is therefore sufficient in line 20 to only add the point labeling a block.

This argument also shows that ω can be added to q only when $r[\omega] = \omega$ gets changed to another point. As this can only happen once, there is a limit on the queue length, which proves that the algorithm terminates. □

Let us now consider what candidates for α we really need for block seeds $\{1, \alpha\}$:

LEMMA 52: Let $1 \in B \subset \Omega$ a block in a block system for G on Ω. Then B is the union of orbits of $\mathrm{Stab}_G(1)$.

Proof: Suppose there is $g \in \mathrm{Stab}_G(1)$ such that $\alpha^g = \beta$. Than for any block B such that $1, \alpha \in B$ we have that $B^g \cap B \neq \emptyset$, thus $B = B^g$. Thus also $\beta \in B$. □

This lemma shows that we do not need to test minimal blocks for all $\alpha \in \Omega$, but that it is sufficient to test those α which are representatives for the orbits of $\mathrm{Stab}_G(1)$ on Ω, and in this case we can actually seed the block with $\{1\} \cup \alpha^{\mathrm{Stab}_G(1)}$.

If we did not yet compute a stabilizer chain for G obtaining $\mathrm{Stab}_G(1)$ is hard. In this case we just approximate $\mathrm{Stab}_G(1)$ by a subgroup U generated by a few random Schreier generators and consider the orbits of U instead.

PERFORMANCE 53: A newer method, providing a better complexity, but also more complicated is described in [Ser03].

Once we have found a block system, the homomorphism representing the action on the blocks is obtained by an easy application of the orbit algorithm.

II.4.4. Wreath Products and the Embedding theorem. In the same way that every intransitive group is a subgroup of a direct product, we want to get an "universal" group containing every imprimitive group.

DEFINITION 54: Let G be a group and H a permutation group, acting on $\Delta = \{1, \dots, n\}$. The *wreath product* of G with H is

$$G \wr H = (\underbrace{G \times \cdots \times G}_{n \text{ times}}) \rtimes H$$

with H acting on $G \times \cdots \times G$ by permuting the components of this direct product. The subgroup $G \times \cdots \times G \lhd G \wr H$ is called the *basis* of the wreath product.

Suppose that G is also a permutation group, acting on Ω. Then $G \wr H$ can be represented as a permutation group acting on n disjoint copies of Ω: Each copy of G in the basis acts on "its" copy of Ω, H is permuting these copies. We call this the *imprimitive action* of $G \wr H$, as the copies of Ω form a nontrivial block system.

THEOREM 55 (KRASNER, KALOUJNINE): Let G be a transitive, imprimitive permutation group. Let $\mathcal{B}$ be a nontrivial block system for G with $1 \in B \in \mathcal{B}$ and let $T = \mathrm{Stab}_G(B)$. Let $\psi\colon G \to S_{\mathcal{B}}$ be the action of G on the blocks, and let $\varphi\colon T \to S_B$ be the action of a block stabilizer on its block.

We pick coset representatives r_j for T in G and define $\widetilde{g_j} \in T$ by $r_j g = \widetilde{g_j} r_{jg}$.

Then there is a monomorphism $\mu\colon G \to T^{\varphi} \wr G^{\psi}$, given by

$$g \mapsto (\widetilde{g_{1g^{-1}}}^{\varphi}, \dots, \widetilde{g_{ng^{-1}}}^{\varphi}; g^{\psi})$$

Proof: We have proven this theorem in M666 in the context of induced representations. □

NOTE 56: There unfortunately is no nice theorem parameterizing all subgroups of wreath products which are imprimitive permutation groups. An algorithm to construct such subgroups was given in [Hul05]

II.4.5. Basic Sylow Subgroup Computation. The following method works reasonably well in practice and also serves as a good example on how algorithms use reductions of intransitivity and imprimitivity. It is, however, not of polynomial time as it uses a backtrack search. A much

We eliminate trivial and redundant ($i = g^{-1}$) generators and get

$$S = \langle c, e, g \mid c^2 = e^2 = cge = (g^{-1})^5 = 1 \rangle$$

We now can eliminate $g = c^{-1}e^{-1}$ and get

$$S = \langle c, e \mid c^2 = e^2 = (ec)^5 = 1 \rangle$$

which is easily seen to be a dihedral group of order 10 (so the initial group G must have had order $6 \cdot 10 = 60$).

NOTE 122: The occurrence of cyclic conjugates of the relator $cgfde$ is not really surprising, but simply a consequence of the power relator $(ab)^5$. One can incorporate this directly in the rewriting algorithm, similarly to a treatment (generator elimination) for relators of length 1.

NOTE 123: A small variant of this rewriting process is the so-called *Modified Todd-Coxeter* algorithm that produces a presentation for S in a given generator set. In general it produces worse presentations than the presentation in Schreier generators obtained here.

As an application we describe a method often used to determine the order of a finite finitely presented group: We enumerate the cosets of a cyclic subgroup $\langle x \rangle$ (often x is itself chosen as a generator of the group). Then we rewrite the presentation to obtain a presentation for $\langle x \rangle$. Then $|G| = [G\colon \langle x \rangle] \cdot |\langle x \rangle|$. Since the subgroup is known to be cyclic, any resulting relator will in effect have the form $x^{e_i} = 1$, thus $|\langle x \rangle| = \mathrm{lcm}_i(e_i)$ can be obtained easily.

PERFORMANCE 124: As the rewritten presentation is on $n(m-1)+1$ generators, even Tietze transformations typically cannot rescue humongous presentations obtained for subgroups of large index. Thus there is a natural limit (a few thousand) on the subgroup index for which rewriting is feasible.

III.9. Abelian Quotients

We have seen already a method (the GQuotient algorithm, algorithm 102) which for a finitely presented group G finds all quotient groups G/N isomorphic to a given finitely presented group H.

In general, one would like to do this not only for a specific H, but for the "largest possible H" within some class of groups. Such algorithms are called "quotient algorithms", we will encounter them again later in section IV.5.

Here we want to determine the largest abelian quotient. By the "quotient subgroup" paradigm, this is equivalent to determining the derived subgroup G' of a finitely presented group $G = \langle \underline{\mathbf{g}} \mid R \rangle$.

coset x and generator g also the appropriate Schreier generator s, such that $t_x \cdot g = s \cdot t_x$. We can construct this from the permutation action on the cosets by a simple orbit algorithm.

We then scan every relator at every coset and collect the occurring Schreier generators.

NOTE 120: In practice, Reidemeister rewriting often produces many trivial Schreier generators and relators that are trivial or of length 1 or 2 (which immediately eliminate generators). Thus typically the resulting presentation is processed by Tietze transformations to eliminate such trivialities.

EXAMPLE 121: Let us go back to example 107 where we enumerated cosets. Our coset table was

	a	a^{-1}	b	b^{-1}
1	1	1	$\underline{2}$	$\underline{3}$
2	$\underline{4}$	4	3	1
3	3	3	1	2
4	2	2	$\underline{5}$	$\underline{6}$
5	6	6	6	4
6	5	5	4	5

with representatives

$$\begin{aligned} t_1 &= 1 \\ t_2 &= b \\ t_3 &= b^{-1} \\ t_4 &= ba \\ t_5 &= bab \\ t_6 &= bab^{-1} \end{aligned}$$

This defines the following nontrivial Schreier generators and the augmented coset table:

$$\begin{aligned} c &= t_1 a t_1^{-1} = a \\ d &= t_2 b t_3^{-1} = b^3 \\ e &= t_3 a t_3^{-1} = b^{-1}ab \\ f &= t_4 a t_2^{-1} = baab^{-1} \\ g &= t_5 a t_6^{-1} = bababa^{-1}b^{-1} \\ h &= t_5 b t_6^{-1} = babbba^{-1}b^{-1} \\ i &= t_6 a t_5^{-1} = bab^{-1}ab^{-1}a^{-1}b^{-1} \end{aligned}$$

and

	a	a^{-1}	b	b^{-1}
1	$c1$	$c^{-1}1$	$\underline{2}$	$\underline{3}$
2	$\underline{4}$	$f^{-1}4$	$d3$	1
3	$e3$	$e^{-1}3$	1	$d^{-1}2$
4	$f2$	2	$\underline{5}$	$\underline{6}$
5	$g6$	$i^{-1}6$	$h6$	4
6	$i5$	$g^{-1}5$	4	$h^{-1}5$

Now we trace the relators at every coset and collect the Schreier generators on the way:

	a^2	b^3	$(ab)^5$
1	c^2	d	$cgfde$
2	f	d	$gfdec$
3	e^2	d	$ecgfd$
4	f	h	$fdecg$
5	gi	h	$gfdec$
6	ig	h	$(ih)^5$

Eliminating duplicates and cyclic permutations (which are just conjugates) we get the presentation

$$S = \langle c, d, e, f, g, h, i \mid c^2 = d = e^2 = f = h = gi = ig = cgfde = (ih)^5 = 1 \rangle$$

more elaborate (polynomial time) algorithm has been proposed by Kantor [Kan85] which however just now is reaching feasibility as it requires a lot of subroutines.

The basic idea of the calculation is that if $\varphi\colon G \to H$ is a homomorphism, we first compute a p-Sylow subgroup $S^\varphi \leq H$, then S must contain a p-Sylow subgroup of G.

LEMMA 57: Suppose G is a subdirect product of $A = G^\alpha$ with $B = G^\beta$. Let $Q \leq G$ be such that Q^α is a p-Sylow subgroup of A and let $\nu = \beta|_Q$ be the restriction of β to Q. Let $P \leq Q$ be such that P^ν is a p-Sylow subgroup of Q^ν. Then P is a p-Sylow subgroup of G.

Proof: Clearly Q contains a p-Sylow subgroup of G and P contains a p-Sylow subgroup of Q. Furthermore P^α and P^β are p-groups, so P is a subdirect product of p-groups. □

We will make use of this lemma in two situations: If G is intransitive (with homomorphisms corresponding to orbit actions) and if G has two different minimal block systems (with action on the blocks as homomorphisms).

If G is imprimitive and has only one one minimal block system with block action φ we can reduce to the situation that G^φ is a p group.

If we cannot reduce further, we use the fact that the p-Sylow subgroup has a trivial center: By random search find an element $h \in G$ such that $p \mid |h|$, then $g = h^{\frac{|h|}{p}}$ is an element of order p. Compute $C := C_G(g)$. Recursively, we compute a p-Sylow subgroup S of C. (As C stabilizes the partition of Ω into orbits of $\langle g \rangle$ it must be imprimitive and we can recurse.) As $g \in Z(C)$, we know that $g \in S$.

If $p \nmid [G{:}C]$, then C must contain a Sylow subgroup and so S is the desired result.

Otherwise there exists a P-Sylow subgroup P of G such that $S \leq P$ and there is $z \in Z(P)$ of order p. Thus z commutes with $g \in S$ and thus $z \in C$. Thus z is contained in a p-Sylow subgroup of C. These groups are all conjugate, thus we can assume without loss of generality that $z \in Z(S)$. We thus search for an element of order p in $Z(S)$ such that $C_G(z)$ contains a p-Sylow subgroup.

ALGORITHM 58: Input: A group G on Ω and a prime p
Output: A p-Sylow subgroup $S \leq G$.
begin
  if G is a p-group then
    return G
  elif G is intransitive on Ω then

```
    recurse on orbit actions, using lemma 57
  elif p ∤ |Ω| then
    recurse on Stab_G(1)
  elif G has two minimal block systems then
    recurse on block actions action, using lemma 57
  elif G has unique minimal block system then
    ensure (recursively) the image of block action of G is a p-group
  fi;
  let h ∈ G such that p | |h| and set g = h^(|h|/p).
  if p^2 ∤ |G| then
    return ⟨g⟩
  fi;
  Let C = C_G(g);
  Recursively, compute a p-Sylow subgroup S of C.
  if p ∤ [G:C] then
    return S;
  fi;
  Let Z = Z(S) {iterative centralizer computation}
  for z ∈ Z, |z| = p do
    C := C_G(z);
    if p ∤ [G:C] then
      recurse on C
    fi;
  od;
end
```

II.5. Primitive Groups

Primitive groups are interesting in several ways: They are the images of the permutation action of a group on cosets of maximal subgroups. By theorem 55 we also know that every transitive group embeds in an (iterated) wreath product of primitive groups.

The marvelous fact now is that primitivity is a strong enough condition to give a rather detailed description of such groups. Indeed this description is strong enough, that it is possible to enumerate primitive groups for rather large degrees – currently this has been done up to degree 2000 [DM88, The97, RDU03].

II.5.1. Some Properties.

LEMMA 59: Let G be a transitive group on Ω and $N \lhd G$. Then the orbits of N form a block system of G

Schreier's theorem 15) rewrite $w(g_1, \ldots, g_m)$ as a word in the Schreier generators $s_{i,j}$. (For this we only need to know the action of G on the cosets of S.)

We form a second free group E on a generating set $\{e_{i,j} \mid (i,j) \in I\}$. Let $\varrho\colon U \to E$ be the *rewriting map*, which for any such word $w(f_1, \ldots, f_m) \in U$ returns a word $\varrho(w) \in E$ which represents the expression of $w(g_1, \ldots, g_m)$ as a word in the nonredundant Schreier generators $s_{i,j}$.

THEOREM 117 (REIDEMEISTER): Let $G = \left\langle \underline{\mathbf{g}} \mid R \right\rangle$ a finitely presented group and $S \leq G$ with $\left[G{:}S\right] < \infty$. Then S is finitely presented and

$$S = \left\langle e_{i,j} \text{ for } (i,j) \in I \mid \varrho(t_x r_y t_x^{-1}), 1 \leq x \leq n, 1 \leq y \leq k \right\rangle$$

is a presentation for S on the Schreier generators. (We are slightly sloppy in the notation here by interpreting the t_x as representatives in F.)

Proof: If we evaluate $t_x^{-1} r_y t_x$ in the generators $\underline{\mathbf{g}}$ of G, these relators all evaluate to the identity. Therefore the rewritten relators must evaluate to the identity in S. This shows that there is an epimorphism from the finitely presented group onto S.

We thus only need to show that any relation among the $s_{i,j}$ can be deduced from the rewritten relators: Let $w(\underline{\mathbf{e}})$ be a word such that $w(\underline{\mathbf{s}}) = 1$ in S. By replacing $e_{i,j}$ by $(t_i f_j \overline{t_i f_j})^{-1}$ we can get this as a new word $v(\underline{\mathbf{f}})$, such that $v(\underline{\mathbf{g}}) = 1$ in G. Therefore v can be expressed as a product of conjugates of elements in R: $v(\underline{\mathbf{f}}) = \prod_z r_{x_z}^{u_z(\underline{\mathbf{f}})}$ where the u_z denote words for the conjugating elements.

Now consider a single factor $r^{u(\underline{\mathbf{f}})}$. As the t_x are representatives for the right cosets of S, we can write $u(\underline{\mathbf{f}}) = t_x^{-1} \cdot q(\underline{\mathbf{f}})$ where $q(\underline{\mathbf{g}}) \in S$ and x is defined by $S \cdot u(\underline{\mathbf{g}})^{-1} = S \cdot t_x$. Thus $r^{u(\underline{\mathbf{f}})} = (t_x \cdot r \cdot t_x^{-1})^{q(\underline{\mathbf{f}})}$. Rewriting this with ϱ, we get $\varrho(t_x \cdot r \cdot t_x^{-1})^{\varrho(q)}$, which is a conjugate of a rewritten relator. □

We note an important consequence, though we will not use it:

COROLLARY 118 (NIELSEN, SCHREIER): Any subgroup of finite index n of a free group of rank m is free of rank $n \cdot (m-1) + 1$.

Proof: Rewriting will produce no relators for the subgroup. □

NOTE 119: This theorem also holds without the "finite index" qualifier. It is usually proven in the general form using coverings.

To perform this rewriting in practice, it is easiest to form an *augmented coset table* by storing (for entries that are not definitions) in position for

NOTE 116: There are variations to only obtain normal subgroups. However, given the knowledge of all small groups up to order 2000, the following approach makes more sense: If $N \lhd G$ has small index consider $Q = G^\varphi = G/N$.

Typically either Q is solvable or even nilpotent (and then N can be found via powerful quotient algorithms) or Q has a faithful permutation representation on the cosets of a subgroup $U \leq Q$ of small index. (Here we use the knowledge of groups of small order to obtain concrete bounds.)

Then the preimage of U under φ can be obtained by an ordinary low-index calculation for such a small index.

An somewhat alternative view of this is to consider the low-index algorithm as a version of the GQuotient algorithm 102. Enumerating all possible columns of the coset table is in effect like enumerating all m-tuples of elements in the symmetric group S_n that fulfill the relations, replacing the "surjectivity" condition to be just transitivity of the image. The principal benefit of the low-index routine is that it implicitly uses the defining relators to impose conditions on the permutations.

III.8. Subgroup Presentations

Any subgroup of finite index of a finitely presented group is finitely generated by lemma 15. In fact it also is finitely presented:

To state the theorem we need some definitions: Let $F = \langle f_1, \ldots, f_m \rangle$ a free group and $R = \{r_1(\underline{\mathbf{f}}), \ldots, r_k(\underline{\mathbf{f}})\}$ a finite set of relators that defines the finitely presented group $G = \left\langle \underline{\mathbf{g}} \mid R \right\rangle$ as a quotient of F. We consider $\underline{\mathbf{g}}$ as the elements of G that are the images of the free generators $\underline{\mathbf{f}}$.

Suppose that $S \leq G$ with $n = \left[G{:}S\right] < \infty$. We choose a transversal $t_1 = 1, t_2, \ldots, t_{\left[G:S\right]}$ for S and form the Schreier generators $s_{i,j} = t_i g_j (\overline{t_i g_j})^{-1}$ for S.

If the coset representative t_i is defined as $t_i = t_j \cdot g_x$, the Schreier generator $s_{j,x}$ is trivial by definition. We call the pair (j, x) "redundant" and let $I \subset \{1, \ldots, n\} \times \{1, \ldots, m\}$ be the set of all index pairs that are not redundant, i.e. the set of Schreier generators that are not trivial by definition is $\{s_{i,j} \mid (i,j) \in I\}$. As there are $n-1$ coset representatives that are defined as image of a "smaller" coset representative under a group generator, we have $|I| = n \cdot m - (n-1) = n \cdot (m-1) + 1$.

As G is a quotient of F, we have a subgroup $U \leq F$ which is the full preimage of S under the natural epimorphism $F \to G$

Now consider that $w(f_1, \ldots, f_m) \in U$ is a word in $\underline{\mathbf{f}}$ such that the corresponding element $w(g_1, \ldots, g_m) \in S$. Then we can (as in the proof of

Proof: Let Δ be an orbit of N and $g \in G$. We need to show that Δ^g is a subset of an orbit. (If this holds and it was not an orbit, we can apply g^{-1} to the enclosing orbit and obtain that Δ was a proper subset of an orbit.) Thus let $\delta^g, \gamma^g \in \Delta^g$ for $\delta, \gamma \in \Delta$. Then there is $n \in N$ such that $\delta^n = \gamma$ and thus $(\delta^g)^{g^{-1}ng} = \gamma^g$. □

COROLLARY 60: If G is primitive on Ω then N must act transitively.

The heart of the analysis will be the consideration of minimal normal subgroups:

LEMMA 61: Let G be a group and $N \lhd G$ a minimally normal subgroup. Then $N \cong T \times \cdots \times T$ with T simple.

Proof: Let $M \lhd N$ be the first proper subgroup in a composition series of N. Then $N/M \cong T$ is simple. Now consider the orbit $\mathcal{M} = M^G$ of M under G. Clearly $K := \bigcap_{S \in \mathcal{M}} S \lhd G$, thus $K := \langle 1 \rangle$. Thus the homomorphism $\varphi\colon N \to \underset{S \in \mathcal{M}}{\times} N/S =: D, g \to (S_1 g, S_2 g, \ldots)$ is faithful and $N \cong N^\varphi$ is a subdirect product of the groups N/S ($S \in \mathcal{M}$). But $N/S \cong T$ is simple, thus the subdirect product degenerates to a direct product. □

DEFINITION 62: The *socle* of a group G is the subgroup generated by all minimal normal subgroups:

$$\mathrm{Soc}(G) = \langle N \lhd G \mid N \text{ is minimally normal} \rangle$$

LEMMA 63: $\mathrm{Soc}(G)$ is the direct product of minimal normal subgroups.

Proof: Let $M \leq \mathrm{Soc}(G)$ be the largest normal subgroup of G within S which is a direct product of minimal normal subgroups. If $M \neq \mathrm{Soc}(G)$ there exists $N \lhd G$, minimally normal, such that $N \not\leq M$. But then $M \cap N \lhd G$. As N is minimally normal this implies that $M \cap N = \langle 1 \rangle$. Thus $\langle M, N \rangle = M \times N \leq \mathrm{Soc}(G)$, contradicting the maximality of M. □

Next we want to show that for a primitive group $\mathrm{Soc}(G)$ is either minimally normal, or the product of two isomorphic minimally normal subgroups:

DEFINITION 64: A permutation group G is *semiregular* on Ω if $\mathrm{Stab}_G(\omega) = \langle 1 \rangle$ for every $\omega \in \Omega$.

Thus G is regular on Ω if and only if G is transitive and semiregular.

LEMMA 65: Let $G \leq S_\Omega$ be transitive. Then $C := C_{S_\Omega}(G)$ is semiregular.

Proof: Suppose that $c \in \mathrm{Stab}_C(\omega)$ and let $\delta \in \Omega$. Then there is $g \in G$ such that $\delta = \omega^g = \omega^{cg} = \omega^{gc} = \delta^c$, thus $c \in \mathrm{Stab}_C(\delta)$ for every δ. Thus $c = 1$. □

LEMMA 66: Let G be a primitive group on Ω. Then one of the following situation holds:

a) $Soc(G)$ is minimally normal
b) $\mathrm{Soc}(G) = N \times M$ with $N, M \lhd G$ minimally normal and $N \cong M$ is not abelian.

Proof: Suppose that $\mathrm{Soc}(G)$ is not minimally normal. By lemma 63 we have that $\mathrm{Soc}(G) = N \times M$ with $N \lhd G$ minimally normal and $\langle 1 \rangle \neq M \lhd G$. Then $M \leq C_G(N)$ and $N \leq C_G(M)$.

As G is primitive, N is transitive on Ω. Thus by lemma 65 we have that M must be semiregular. On the other hand $M \lhd G$ implies that M is also transitive on Ω, thus N is semiregular. In summary thus both N and M must be regular, and thus $|N| = |\Omega| = |M|$.

For $n \in N$ there exists a unique element $m_n \in M$ such that $(1^n)^m = 1$. Let $\varphi\colon N \to M$ given by $n \mapsto m_n$. Then φ is clearly a bijection. Furthermore (using that $M, N \leq S_\Omega$) for $k, n \in N$:

$$1^{k \cdot n \cdot m_k \cdot m_n} = 1^{k \cdot m_k \cdot n \cdot m_n} = \left((1^k)^{m_k}\right)^{n \cdot m_n} = 1^{n \cdot m_n} = 1$$

and therefore $m_k m_n = m_{kn}$. Thus φ is an isomorphism.

If N was abelian, then $N \times M$ is abelian and transitive, thus $|N \times M| = |\Omega|$, contradiction. □

We thus have that $\mathrm{Soc}(G) \cong T \times \cdots \times T$ with T simple. We say that $\mathrm{Soc}(G)$ is *homogeneous of type* T.

THEOREM 67: Let G be primitive on Ω and $\mathrm{Soc}(G)$ abelian. Then $|\Omega| = p^m$ for some prime p and $G = \mathrm{Soc}(G) \rtimes \mathrm{Stab}_G(1)$ with $\mathrm{Stab}_G(1)$ acting (by conjugation) irreducibly and faithfully on $\mathrm{Soc}(G) \cong \mathbb{F}_p^m$.

Proof: If $\mathrm{Soc}(G)$ is abelian, it is minimally normal and thus $\mathrm{Soc}(G) \cong \mathbb{F}_p^m$. It must act regularly, thus $|\Omega| = p^m$.

Now consider $S := \mathrm{Stab}_G(1)$. Clearly $\mathrm{Soc}(G) \not\leq S$. As $S < G$ is a maximal subgroup we thus have that $G = \mathrm{Soc}(G)S$. As $\mathrm{Soc}(G)$ is abelian, $S \cap \mathrm{Soc}(G) \lhd \mathrm{Soc}(G)$. Also $S \cap \mathrm{Soc}(G) \lhd S$. Thus $S \cap \mathrm{Soc}(G) \lhd G$ and therefore $S \cap \mathrm{Soc}(G) = \langle 1 \rangle$, thus G is a semidirect product.

If S was not acting irreducibly on $\mathrm{Soc}(G)$ let $T \leq \mathrm{Soc}(G)$ be a proper submodule. Then T is normalized by S and $T \lhd \mathrm{Soc}(G)$, thus $T \lhd G$ contradicting the fact that $\mathrm{Soc}(G)$ is minimally normal.

The kernel of the action of S on $\mathrm{Soc}(G)$ is contained in $C_G(\mathrm{Soc}(G)) = \mathrm{Soc}(G)$, thus the action is faithful. □


```
4: if no coincidence arose then
5:     call DESCENDANTS(S);
6: fi;
end
```

Next we want to consider conjugacy. Using a standard approach to construction of objects up to a group action, we define a (somehow arbitrary) order on coset tables which tests conveniently for partial coset tables, and then for each (partial) table test whether it can be the smallest in its class (the group acting by conjugation of subgroups in our case). If not we discard the candidate.

Such a test would be performed before line 5 of the function TRY.

The ordering of coset tables we use is lexicographic, considering the table row by row. I.e. for two tables S, T of size n we have that $T < S$ if for some $1 \leq x \leq n$ and some generator g the following holds:

- For all $y < x$ and any generator h, we have that y^h is the same in S and T
- For all generators h before g we have that x^h is the same in S and T
- x^g is smaller in T than in S.

To determine the coset table for a conjugate, observe that a coset table yields the conjugation action on the cosets of a subgroup. In this action the subgroup is stabilizer of the point 1, and every conjugate is stabilizer of another point x. If $g \in G$ is an element such that $1^g = x$, then g would conjugate the subgroup to the conjugate corresponding to x. Among coset tables, this conjugation would happen as relabeling of points and permutation of cosets by g.

But (the permutation action *is* given by the coset table) we can determine such an element g from the coset table.

As we have only a partial coset table this construction may not yet succeed (or we may be lacking entries to compare yet), in any case it will eliminate groups that are not first in their class. We also often can perform this pruning already for an only partially constructed coset table.

PERFORMANCE 115: There are many variations and improvements. For example, as long relators rarely yield a deduction but only are conditions to test, it can make sense to only consider the shorter (whatever this means) relators for the determination of coset tables and simply test each table obtained afterward for the remaining relators.

ALGORITHM 114: This is a basic version of the low index algorithm without elimination of conjugates.

Input: $G = \langle \mathbf{G} \mid R \rangle$, index n

Output: All subgroups of G of index up to n, given by coset tables.

begin

Initialize T as empty table for G.

$L := []$;

return DESCENDANTS(T).

end

The DESCENDANTS routine performs the actual assignment and calls a second routine TRY to verify validity and process deductions. We assume that an image 0 indicates that the image is not yet defined

DESCENDANTS(T)

begin

1: if T is complete then
2: Add T to L;
3: else
4: m=number of cosets defined in T
5: Let coset x under generator g be the first undefined ($x^g = 0$) image.
6: for $y \in [1..m]$ do
7: if $y^{g^{-1}} = 0$ then {otherwise we have an image clash}
8: Let S be a copy of T;
9: In S set $x^g = y$ and $y^{g^{-1}} = x$;
10: TRY(S,x,g);
11: fi;
12: od;
13: if $m < n$ then {is one more coset possible?}
14: Let S be a copy of T
15: Add coset $m+1$ to S;;
16: In S set $x^g = m+1$ and $(m+1)^{g^{-1}} = x$;
17: TRY(S,x,g);
18: fi;
19: fi;

end

The validity test is the last routine: TRY(S, x, g)

Input: A partial coset table S in which an assignment x^g has just been made

Output: Perform dependencies and continue search if no coincidences arise

begin

1: Empty the deduction stack;
2: Push x, g on the deduction stack;
3: Process deductions for S as in coset enumeration;

It is easily seen that vice versa any such semidirect product acts primitively on $\mathbb{F}_p^m$.

COROLLARY 68: Let G be a solvable group and $M < G$ a maximal subgroup. Then $[G{:}M] = p^m$ for a prime p.

Proof: The image of the action of G on the cosets of M is a primitive group with an abelian minimal normal subgroup. □

LEMMA 69: Let G be a group such that $Z(\mathrm{Soc}(G)) = \langle 1 \rangle$. Then $G \leq \mathrm{Aut}(\mathrm{Soc}(G))$.

Proof: Consider the action of G by conjugation on $\mathrm{Soc}(G)$. The kernel is $C_G(\mathrm{Soc}(G)) \lhd G$. A minimal normal subgroup contained in $C_G(\mathrm{Soc}(G))$ would be in $Z(\mathrm{Soc}(G))$, thus this action is faithful. □

LEMMA 70: If $N = \underbrace{T \times \cdots \times T}_{m}$ with T non-abelian simple, then $\mathrm{Aut}(N) = \mathrm{Aut}(T) \wr S_m$

Proof: Let T_i be the i-th direct factor. Let $\varphi \in \mathrm{Aut}(N)$. Let $R := T_1^\varphi$. Then $R \lhd N$. Consider some nontrivial element of R as element of a direct product $r = (t_1, \ldots, t_m)$ with $t_i \in T_i$. Suppose that $t_j \neq 1$ for some j. As $Z(T_j) = \langle 1 \rangle$ there exists $y \in T_j$ such that that $t_j^y \neq t_j$. Set $s := r^y / r \neq 1$. Then $s \in R$ and $s \in T_j$. As T_j is simple and $R \lhd N$ we thus get that $T_j \leq R$, thus $R = T_j$.

This shows that every automorphism of N permutes the T_i. An automorphism that fixes all T_i then must act on every T_i as an element of $\mathrm{Aut}(T_i) = \mathrm{Aut}(T)$. □

COROLLARY 71: Let G be primitive with $\mathrm{Soc}(G)$ non-abelian of type T. Then we can embed $G \leq Aut(T) \wr S_m$.

II.5.2. Types. We now want to describe some important classes of primitive groups:

The first class is a different action of wreath products: Let G be a permutation group on Ω and H a permutation group on Δ. So far we have had the wreath product $W := G \wr H$ act (imprimitively) on $\Omega \times \Delta$. We now define a different action of W on Ω^Δ. This is a much larger set, surprisingly, the action will turn out to be primitive in many cases.

The action is easiest described if (assume that $|\Delta| = d$) we consider Ω^Δ as a d-dimensional cube each side of which is labeled by Ω. We then let $G \times \cdots \times G$ act independently in each dimension and H permute the

dimensions. That is we define

$$(\omega_1,\ldots,\omega_d)^{(g_1,\ldots g_d;h)} := (\omega_{1'}^{g_{1'}},\ldots,(\omega_{d'}^{g_{d'}}) \qquad \text{with} \qquad i' = i^{h^{-1}}.$$

an easy, but tedious calculation shows that this indeed is a group action, which is called the *product action*. We note that in this action the base group G^d acts transitively.

THEOREM 72: Suppose that Ω, Δ are finite. Then $G \wr H$ in the product action is primitive if and only if G is primitive, but nonregular on Ω and H transitive on Δ.

NOTE 73: We do not require G to be "minimal" in this theorem. Essentially, using the fact that $(A \wr B) \wr C = A \wr (B \wr C)$, we could enforce this by increasing H.

For the second class of examples, consider a socle of the form $T^m = T \times \cdots \times T$ with T simple. Let D be a diagonal subgroup $\{(t,t,\ldots,t) \mid t \in T\}$. We consider the action of the socle on the cosets of D (of degree $n = |T|^{m-1}$).

As we want to extend this permutation action to a primitive group, we next consider the normalizer $N = N_{S_n}(T^m)$. Clearly all elements of N induce automorphisms of T^m, however the following lemma show that not all automorphisms can be realized:

LEMMA 74: Let $G \leq S_n$ be a transitive group and $\varphi \in \mathrm{Aut}(G)$. Then φ is induced by $N_{S_n}(G)$ (i.e. here exists $h \in S_n$ such that $g^\varphi = g^h$ for every $g \in G$, obviously $h \in N_{S_n}(G)$ in this case) if and only if $\mathrm{Stab}_G(1)^\varphi = \mathrm{Stab}_G(j)$ for some $1 \leq j \leq n$.

Proof: Homework □

Using this lemma, one can show that not all elements of $\mathrm{Aut}(T) \wr S_m$ are induced by permutations, in fact outer automorphisms need to act simultaneously on all components in the same way. Thus $N = T \wr S_m . \mathrm{Out}(T) = T^m \rtimes (S_m \times \mathrm{Out}(T))$ with the outer automorphisms acting simultaneously on all components. A group $T^m \leq G \leq N$ is said to be of *diagonal type*.

THEOREM 75: A group of diagonal type is primitive if $m = 2$ or the action of G on the m copies of T is primitive.

II.5.3. The O'Nan-Scott Theorem. We now can state a theorem that classifies the structure of all primitive groups. The theorem was stated first (with a small error) by L. SCOTT in 1979. (In principle it would have been possible to prove this theorem 50 years earlier, but the reduction to the simple case only made sense with the classification of the finite simple

A standardized coset table thus is the coset table we would obtain if we performed a pure Felsch-style enumeration and after each coincidence relabeled cosets to avoid "gaps".

If we have a coset table we can easily bring it into standard form by running through cosets and withing cosets through generator images and reassigning new labels according to the order in which cosets appear.

The following lemma now is obvious:

LEMMA 113: There is a bijection between subgroups of G of index n and standardized coset tables for G with n cosets.

III.7. Low Index Subgroups

A prominent variation of coset enumeration is the so-called *Low-Index* algorithm that for a given n will find all subgroups of a finitely presented group $G = \left\langle \mathbf{g} | R \right\rangle$ of index $\leq n$ (up to conjugacy).

We will construct these subgroups by constructing all valid standardized coset tables for G on up to n cosets.

For simplicity let us initially assume that we do not want to eliminate conjugates:

The basic step in the algorithm (computing one descendant) takes a partially completed, standardized, coset table, involving $k \leq n$ cosets. (The initialization is with the empty coset table.) If the table is in fact complete, it yields a subgroup.

Otherwise we take the next (within cosets and within each coset in order of generators) open definition, say the image of coset x under generator g.

We now split up in several possibilities on assigning this image: We can assign x^g to be one of the existing cosets $1,\ldots,k$, or (if $k < n$) a new coset $k+1$.

For each choice we take a copy of the coset table and make in this copy the corresponding assignment. Next we run the deduction check, as in ordinary coset enumeration. (According to remark 109, we only need to scan the relators in R_g^c at coset x and $R_{g^{-1}}^c$ at x^g, as well as relators for consequential deductions.) We enter deductions in the table. However if a coincidence occurs, we know that we made an invalid choice, and abandon this partial table, backtracking to the next (prior) choice.

Otherwise we take this new partial table (which so far fulfills all relators as far as possible and is standardized by the way we selected the next open definition) and compute its further descendants.

More formally, this gives the following algorithm:

Felsch: (1959/60) Define the next coset as the first open entry (by rows and within a row by columns) in the coset table. This guarantees that the image of each coset under each generator will be defined at some point.

HLT: (1953, for Haselgrove[1], Leech and Trotter). If there are gaps of length 1 in a subgroup or relator table, fill these gaps (in the hope of getting immediately a consequence). This method is harder to understand theoretically, but often performs better in practice.

There is a large corpus of variants and modifications to these strategies (for example the addition of redundant relators). In particular with hard enumerations often just particular variants will finish.

THEOREM 111 (Mendelssohn, 1964): Suppose that $[G:S] < \infty$ and that the strategy used guarantees that for every defined coset a and every generator g the images a^g, and $a^{g^{-1}}$ will be defined after finitely many steps, then the coset enumeration terminates after finitely (but not bounded!) many steps with the correct index.

Proof:(Idea) If the process terminates, the columns yield valid permutations for the action on the cosets of S. To show termination, assume the contrary. By the condition we can (by transfinite induction) build an *infinite* coset table which would contradict $[G:S] < \infty$. □

III.6.3. Applications and Variations. A principal use of coset enumeration is to get a quotient representation for subgroups for purposes such as element tests or subgroup intersection. We will also see in section III.8 that the coset tables themselves find use in the calculation of subgroup presentations.

One obvious application is the size of a group, by enumerating the cosets of the trivial subgroup. (However in practice one enumerates modulo a cyclic subgroup and obtains a subgroup presentation.)

In general there are many different coset tables corresponding to one subgroup which simply differ by the labels given to the different cosets. For comparing coset tables or processing them further it can be convenient to relabel the cosets to bring the table in a "canonical" form.

DEFINITION 112: A coset table is *standardized* if when running thorough the cosets and within each coset through the generator images (ignoring generator inverses), the cosets appear in order of the integers $1, 2, 3, \ldots$.

[1]Indeed with "s", not with "z"

groups.) He notes that a similar theorem was obtained by M. O'NAN, thus the name.

THEOREM 76 (O'NAN-SCOTT theorem): Let G be a group which acts primitively and faithfully on Ω with $|\Omega| = n$. Let $H = \mathrm{Soc}(G)$ and $\omega \in \Omega$. Then $H \cong T^m$ is homogeneous of type T for T simple and exactly one of the following cases holds.

1. "Affine". T is abelian of order p, $n = p^m$ and $\mathrm{Stab}_G(\omega)$ is a complement to H which acts irreducibly on H.
2. "Almost simple". $m = 1$ and $H \lhd G \leq \mathrm{Aut}(H)$.
3. "Diagonal type". $m \geq 2$ and $n = |T|^{m-1}$. Further, G is a subgroup of $V = (T \wr S_m)$. $\mathrm{Out}(T) \leq \mathrm{Aut}(T) \wr S_m$ in diagonal action and either
   a) $m = 2$ and G acts intransitively on $\{T_1, T_2\}$ or
   b) $m \geq 2$ and G acts primitively on $\{T_1, \ldots, T_m\}$.
   In case a) T_1 and T_2 both act regularly. Moreover, the point stabilizer V_ω of V is of the form $\mathrm{Diag}(\mathrm{Aut}(T)^{\times m}).S_m \cong \mathrm{Aut}(T) \times S_m$ and thus $H_\omega = \mathrm{Diag}(T^{\times m})$.
4. "Product type". $m = rs$ with $s > 1$. We have that $G \leq W = A \wr B$ and the wreath product acts in product action with A acting primitively, but not regularly, on d points and B acting transitively on s points. Thus $n = d^s$. The group A is primitive of either
   a) type 3a with socle T^2 (i.e. $r = 2$, $s < m$),
   b) type 3b with socle T^r (i.e. $r > 1$, $s < m$) or
   c) type 2 (i.e. $r = 1$, $s = m$).
   We have that $W_\omega \cap A^s \cong A_1^{\times s}$ and $\mathrm{Soc}(G) = \mathrm{Soc}(W)$. Furthermore $W = A^{\times s}G$.
5. "Twisted wreath type". H acts regularly and $n = |T|^m$. G_ω is isomorphic to a transitive subgroup of S_m. The normalizer $N_{G_\omega}(T_1)$ has a composition factor isomorphic to T. Thus, in particular, $m \geq k + 1$ where k is the smallest degree of a permutation group which has T as a composition factor.

NOTE 77: We do not discuss the twisted wreath case in detail, but note that the minimum degree for this is 60^6.

The proof of this theorem is not extremely hard (see for example [DM96]), but would take us about 3 lectures.

NOTE 78: There are various versions of this theorem in the literature which in particular differ by the labeling of the cases and sometimes split cases slightly differently. Our version follows [DM96] and in particular [EH01].

The following table gives translations of the labellings used.

Type	1	2	3a	3b	4a	4b	4c	5
[HEO05, Sec.10.1.3]	(i)	(ii)a, d=1	(ii)b	(iii)	(ii)b	(iii)	(ii)b	(ii)c
[DM96, Sec.4.8]	i	iii	iv	iv	v	v	v	ii
[LPS88]	I	II	IIIa	IIIa	IIIb	IIIb	IIIb	IIIc
[Neu86]	I	V	II	III	II	III	IV	IV

Note that for case 3a/b) we change the case distinction of [DM96, Theorem 4.5A] from degree $2/>2$ to intransitive/primitive.

II.5.4. Maximal subgroups of the Symmetric Group. Most classes in the O'Nan-Scott theorem contain obvious maximal elements. Every primitive group thus is contained in such a maximal element.

As one can show that these subgroups are not just maximal in their classes, but also maximal in the symmetric group, we get a classification of maximal subgroups of the symmetric group:

THEOREM 79: Let $M \leq S_n$ be a maximal subgroup of the symmetric group. Then M is permutation isomorphic to one of the following groups:

- A_n
- $S_a \times S_b$ with $a + b = n$.
- $S_l \wr S_m$ in imprimitive action for $lm = n$.
- $AGL_m(p)$ with $n = p^m$
- $S_a \wr S_b$ in product action for $n = a^b$
- $T^a.(S_a \times \mathrm{Out}(T))$ with T simple and $n = |T|^{a-1}$
- $T \leq G \leq \mathrm{Aut}(T)$ for a simple group T

NOTE 80: For some degrees there are inclusions among elements in these classes, but these occur very rarely. A full classification is given in [LPS87].

II.6. Computing a Composition Series

The basic idea of finding a composition series in a permutation group is very easy:

> Given a permutation group G, prove that G is simple or find a homomorphism (which we can evaluate) $\varphi\colon G \to H$ such that H is a permutation group of degree not larger than G and that $N := \mathrm{Kern}\,\varphi > \langle 1 \rangle$.

If we can solve this problem, we can recursively attack N and $G^\varphi \cong G/N$ until we end up with simple composition factors. Pulling the factors of G/N back through φ yields a composition series.

If G is an intransitive permutation group we can take for φ the action of G on one orbit. If G is imprimitive we can take for φ the action on a

a a
1 1 1
2 3 2
3 2 3
4 6 4
5 5
6 4 6

b b b
1 2 1
2 1 2
3 4 5 3
4 5 3 4
5 3 4 5
6 6

a b a b a b a b a b
1 1 2 3 4 6 1
2 3 4 6 1 1 2
3 2 5 3
4 6 1 1 2 3 4
5 6 4 5
6 4 5 6

$b^{-1}ab$
1 1

We now define $1^{b^{-1}} = 7$ and get (from $b^3 : 7 \to 1 \to 2 \to 7$) that also $2^b = 7$. Furthermore, it lets us fill the second subgroup table:

b^{-1}	a	b
1	7!7	1

and therefore $7^a = 7$ and similarly $7^{a^{-1}} = 7$. Thus we get

	a	a^{-1}	b	b^{-1}
1	1	1	2	7
2	3	3	7	1
3	2	2	4	5
4	6	6	5	3
5			3	4
6	4	4		
7	7	7	1	2

a a
1 1 1
2 3 2
3 2 3
4 6 4
5 5
6 4 6
7 7 7

b b b
1 2 7 1
2 7 1 2
3 4 5 3
4 5 3 4
5 3 4 5
6 6
7 1 2 7

a b a b a b a b a b
1 1 2 3 4 6!3 2 7 7 1
2 3 4 6!3 2 7 7 1 1 2
3 2 7 7 1 1 2 3 4!5 3
4 6!3 2 7 7 1 1 2 3 4
5 6 4 5
6 4 5 6
7 7 1 1 2 3 4!5 3 2 7

with the implications $6^b = 3$ and $4^a = 5$. As we had $4^a = 6$, cosets 5 and 6 must coincide. (As we had $6^b = 3$ and $5^b = 3$ there is no subsequent coincidence.)

After this coagulation the table is again closed:

	a	a^{-1}	b	b^{-1}
1	1	1	2	7
2	3	3	7	1
3	2	2	4	5
4	5	5	5	3
5	4	3	3	4
7	7	7	1	2

III.6.2. Strategies. As the examples show, the performance of coset enumeration depends crucially on the definition sequence (i.e. which cosets are defined as what images at what point). A large body of literature exists that outlines experiments and strategies. The two main strategies, named after their initial proposers are:

To avoid having to scan through all relators and all cosets, the following observation is useful: After a definition (or deduction) occurs obviously only relators that make use of this new definition are of interest for renewed checking. Suppose that $abcd$ is a relator, which scans at coset x only up to ab but hangs at c (ignoring the scan from the right). Then a new result can occur only if the coset x^{ab} (meaning the coset if we apply ab to coset x) gets its image under c defined.

In this situation we can instead consider the (equivalent) relator $abcd^{ab} = cdab$ and consider it being scanned starting at coset x^{ab}.

We therefore perform the following preprocessing: We form a list of all cyclic permutations of all relators and their inverses and store these according to the first letter occurring in the permuted relator. Let R_g^c be set set of all such permutations that start with the letter g.

Then if the image of coset y under generator g is defined as z, we scan all relators in R_g^c starting at the coset y and all relators in $R_{g^{-1}}^c$ starting at $z = y^g$.

This then will take care of any relator that might have scanned partially before and will scan further now.

III.6.1. Coincidences. There is one other event that can happen during coset enumeration. Closing a table row might imply the equality of two (prior considered different) cosets. In this case we will identify these cosets, deleting one from the tables. In doing this identification, (partially) filled rows of the coset table might imply further coincidences. We thus keep a list of coincidences to process and add to it any such further identifications and process them one by one.

EXAMPLE 110: For the same groups as in the previous example, suppose we would have followed a different definition sequence, and doing so ended up with the following tables. (The underlined numbers indicate the definition sequence.)

	a	a^{-1}	b	b^{-1}
1	1	1	$\underline{2}$	
2	3	3		1
3	2	2	$\underline{4}$	$\underline{5}$
4	$\underline{6}$	6	5	3
5			3	4
6	4	4		

nontrivial block system. (Either these actions already yield a nontrivial kernel, or they yield a group of smaller degree for which we try again.)

Thus what remains to deal with is the case of G primitive and for such groups the O'Nan-Scott theorem provides structure information. Our main aim will be to find $\mathrm{Soc}(G)$. Then the action of G on $\mathrm{Soc}(G)$ or on the components of $\mathrm{Soc}(G)$ yields homomorphisms with nontrivial kernel.

II.6.1. The affine case. The first case we want to treat is the case of G being primitive affine. In this case the socle is an elementary abelian regular normal subgroup, often abbreviated as EARNS. Given G, we therefore want to find an EARNS in G if it exists. The algorithm for this is due to [Neu86].

Clearly we can assume that G is a group of prime-power degree $n = p^m = |\Omega|$.

We first consider two special cases:

If $\mathrm{Stab}_G(\alpha) = 1$ for $\alpha \in \Omega$, then G is regular, and thus of prime order. It is its own EARNS.

DEFINITION 81: A transitive permutation group G on Ω is called a *Frobenius group* if for every $\alpha, \beta \in \Omega$ ($\alpha \neq \beta$) the two-point stabilizer $\mathrm{Stab}_G(\alpha, \beta) = \langle 1 \rangle$.

A classical (1902) result of Frobenius shows that in our situation G must have an EARNS (for a proof see [Pas68]). As $|G| \leq n(n-1)$ this is easily found. (See [Neu86] for details.)

No suppose we are in neither of these cases. Let $\alpha, \beta \in \Omega$. We consider the two-point stabilizer $G_{\alpha\beta} := \mathrm{Stab}_G(\alpha, \beta) \neq \langle 1 \rangle$. By choosing β from an orbit of $\mathrm{Stab}_G(\alpha)$ of shortest length, we can assume that $G_{\alpha\beta}$ is as large as possible.

Let $\Delta = \{\omega \in \Omega \mid \omega^g = \omega \forall g \in G_{\alpha\beta}\}$. If G has an EARNS N, then for $\gamma \in \Delta$ there is a unique $n \in N$ such that $\alpha^n = \gamma$. Then for $h \in G_{\alpha\beta}$ we have that

$$\underbrace{h^{-1}n^{-1}h}_{\in N \lhd G} n = \underbrace{h^{-1}}_{\in G_{\alpha\beta} \leq \mathrm{Stab}_G(\gamma)} \cdot \underbrace{n^{-1}hn}_{\in \mathrm{Stab}_G(\gamma)} \in N \cap \mathrm{Stab}_G(\gamma) = \langle 1 \rangle$$

and thus $n \in C := C_G(G_{\alpha\beta})$.

In particular C acts transitively on Δ, $N \cap C$ acts regularly on Δ, thus $|\Delta|$ must be a p-power (if either of this is not the case, G has no EARNS).

Now consider the homomorphism $\varphi\colon C \to S_\Delta$, then the image C^φ is transitive on Δ. The kernel of φ consists of elements that stabilize all points in Δ (in particular α and β) and centralize $G_{\alpha\beta}$, thus $\mathrm{Kern}\,\varphi \leq Z(G_{\alpha\beta})$.

For $\gamma \in \Delta \setminus \{\alpha\}$ we have that $G_{\alpha\beta} \leq \mathrm{Stab}_G(\alpha, \gamma)$, but as β was chosen to yield a maximal stabilizer, we get equality. Thus $\mathrm{Stab}_C(\alpha\gamma)^\varphi = \langle 1 \rangle$ and C^φ is a Frobenius group.

Let $K \leq C$ such that $K^\varphi = (N \cap C)^\varphi$ is the EARNS of C^φ. Thus $N \cap K \neq \langle 1 \rangle$ and we just need to get hold of some element in $N \cap K$ to find N.

We claim that K is abelian: This is because K is generated by $\operatorname{Kern}\varphi \leq Z(G_{\alpha\beta})$ and by elements of $N \cap C$ which commute with each other (N is abelian) and with $\operatorname{Kern}\varphi \leq G_{\alpha\beta}$ (as they are in C).

Next compute $P = \{x \in \operatorname{Kern}\varphi \mid x^p = 1\}$. (As $\operatorname{Kern}\varphi$ is abelian, this is easy.)

We now find $x \in K \setminus \operatorname{Kern}\varphi$ such that $|x| = p$.

Then $1 \neq x^\varphi = g^\varphi$ for some $g \in N \cap K \leq$ and $x^{-1}g \in \operatorname{Kern}\varphi$. As K is abelian $\left|x^{-1}g\right| = p$, thus $x^{-1}g = h \in P$ and $g \in N \cap Px$.

We thus run through the elements $h \in P$, and test whether $\langle xh \rangle_G$ is abelian and regular – if so it is the EARNS.

NOTE 82: A variant of this method can be used to find in a regular normal subgroup of G also for type 5 groups.

NOTE 83: Variants of the method can be used to find the largest normal p-subgroup $O_p(G) \lhd G$ and the *radical* of $O_\infty(G) \lhd G$, which is the largest solvable subgroup of G. These methods also construct a homomorphism from G to a group of not larger degree such that the kernel is $O_p(G)$, respectively $O_\infty(G)$

II.6.2. Finding the socle and socle components. The following methods follow [Neu87]. They differ from what is used in practice but give an idea of the methods and are easier to describe:

THEOREM 84 (Schreier's conjecture): Let G be a simple non-abelian group. Then $\operatorname{Aut}(G)/G$ is solvable of derived length ≤ 3.

Proof: Inspection, following the classification of finite simple groups. □

LEMMA 85: Let G be a primitive group with no nontrivial abelian normal subgroup. Let $S := \operatorname{Soc}(G) = T_1 \times \cdots \times T_m$ with $T_i \cong T$ simple non-abelian. Let $U \leq G$ be a 2-Sylow subgroup and $N = \langle Z(U) \rangle_G$. Then $S = N'''$.

Proof: By Feit-Thompson $2 \mid |T_i|$. As T_i is subnormal in G, we know that $U \cap T_i \neq \langle 1 \rangle$. Thus every element of $Z(U)$ must centralize some elements in T_i. Considering G embedded in $\operatorname{Aut}(T) \wr S_m$ we thus see that elements of $Z(U)$ may not move component i. Thus $Z(U) \leq \operatorname{Aut}(T)^m \cap G \lhd G$. Thus $\langle Z(U) \rangle_G \leq \operatorname{Aut}(T)^m \cap G$. But $(\operatorname{Aut}(T)^m)''' = T^m$ by theorem 84.

On the other hand, $Z(U \cap T_i) \neq \langle 1 \rangle$ and (as those elements commute with all other T_j and with $U \cap T_i$, we have that $Z(U \cap T_i) \leq Z(U)$. Thus $Z(U) \cap T_i \neq \langle 1 \rangle$, which shows that $T^m \leq \langle Z(U) \rangle_G$. □

As there is no deduction, we define $4^{b^{-1}} = 6$.

	a	a^{-1}	b	b^{-1}
1	1	1	2	3
2	4	4	3	1
3	3	3	1	2
4	2	2	5	**6**
5				4
6			**4**	

	a	a
1	1	1
2	4	2
3	3	3
4	2	4
5	5	5
6	6	6

	b	b	b
1	2	3	1
2	3	1	2
3	1	2	3
4	5	!**6**	4
5	!**6**	4	5
6	6	**4**	**5**!6

	a	b	a	b	a	b	a	b	a	b
1	1	1	2	4	5	!**6**	4	2	3	3	1
2	4	5	!**6**	4	2	3	3	1	1	2	
3	3	3	1	1	2	4	5	!**6**	4	2	3
4	2	3	3	1	1	2	4	5	!**6**	4	
5							**6**	4	4	5	
6										6	

We conclude $5^a = 6$ and $5^b = 6$. The second table then implies $6^a = 5$. (Also all relator tables are filled with no new deduction.)

	a	a^{-1}	b	b^{-1}
1	1	1	2	3
2	4	4	3	1
3	3	3	1	2
4	2	2	5	6
5	**6**	**6**	**6**	4
6	**5**	**5**	4	**5**

At this point all places in the table are closed and no deductions pending.

Once we have reached the point of all tables closed and no deductions pending, the columns of the coset table give permutation images for the group generators that are consistent with all relators (as we maintained the relator tables).

If there are n rows, we have thus obtained a homomorphism $\varphi\colon G \to S_n$, such that $S^\varphi = \operatorname{Stab}_{G^\varphi}(1)$. This is all we need to represent S as a quotient subgroup. (In particular, $\left[G{:}S\right] = n$ equals the number of rows in the table.)

In the example we would have $a \mapsto (2,4)(5,6)$ and $b \mapsto (1,2,3)(4,5,6)$. We can also read off coset representatives as follows:

$$\begin{aligned} 2 &= 1^b \\ 3 &= 1^{b^{-1}} \\ 4 &= 1^{ba} \\ 5 &= 1^{bab} \\ 6 &= 1^{bab^{-1}} \end{aligned}$$

NOTE 109: On the computer, we can save space by not storing images under inverses and the subgroup and relator tables – we can simply compute their contents by *scanning* through relators (i.e. looking at images of cosets under subsequent generators within the relator *forwards and backwards*), respectively by looking up images.

Define $3 = 1^{b^{-1}}$:

	a	a^{-1}	b	b^{-1}
1	1	1	$\underline{2}$	$\underline{\mathbf{3}}$
2				1
3			**1**	

a a		b b b			a b $ababa$ b a b						b^{-1} a b	
1	1	1	2!**3**	1	1	1	2			**3**	1	1	**3**!**3**	1
2		2	2!**3**	1 2	2				**3**	1	1	2		
3		3	3 **1**	**2**!3	3						3			

We conclude that $2^b = 3$ and $3^a = 3$ (and now also retire the second subgroup table).

	a	a^{-1}	b	b^{-1}
1	1	1	$\underline{2}$	$\underline{3}$
2			**3**	1
3	**3**	**3**	1	**2**

a a			
1	1	1	
2		2	
3	**3**!3		

b b b			
1	2	3	1
2	3	1	2
3	1	2	3

a b a b a b a b a b									
1	1	2					**2**	**3**	3	1
2					**2**	**3**	3	1	1	2
3	**3**	**1**	**1**	**2**					**2**	3

There is no new conclusion. We set $2^a = 4$

	a	a^{-1}	b	b^{-1}
1	1	1	$\underline{2}$	$\underline{3}$
2	$\underline{\mathbf{4}}$		3	1
3	3	3	1	2
4		**2**		

a a			
1	1	1	
2	**4**!2		
3	3	3	
4!**2**		4	

b b b			
1	2	3	1
2	3	1	2
3	1	2	3
4			4

a b a b a b a b a b									
1	1	2	**4**				2	3	3	1
2	**4**				2	3	3	1	1	2
3	3	1	1	2	**4**				2	3
4										4

We conclude $4^a = 2$

	a	a^{-1}	b	b^{-1}
1	1	1	$\underline{2}$	$\underline{3}$
2	$\underline{4}$	**4**	3	1
3	3	3	1	2
4	**2**	2		

a a			
1	1	1	
2	4	2	
3	3	3	
4	2	4	

b b b			
1	2	3	1
2	3	1	2
3	1	2	3
4			4

a b a b a b a b a b									
1	1	2	4			**4**	2	3	3	1
2	4			**4**	2	3	3	1	1	2
3	3	1	1	2	4			**4**	2	3
4	**2**	**3**	**3**	**1**	**1**	**2**	**4**			4

Now we set $4^b = 5$:

	a	a^{-1}	b	b^{-1}
1	1	1	$\underline{2}$	$\underline{3}$
2	$\underline{4}$	4	3	1
3	3	3	1	2
4	2	2	$\underline{\mathbf{5}}$	
5				**4**

a a			
1	1	1	
2	4	2	
3	3	3	
4	2	4	
5		5	

b b b			
1	2	3	1
2	3	1	2
3	1	2	3
4	**5**		4
5		**4**	5

a b a b a b a b a b									
1	1	2	4	**5**		4	2	3	3	1
2	4	**5**		4	2	3	3	1	1	2
3	3	1	1	2	4	**5**		4	2	3
4	2	3	3	1	1	2	4	**5**		4
5								**4**	**4**	5

Using this lemma we easily obtain $\mathrm{Soc}(G)$.

Note that the only case in which $\mathrm{Soc}(G)$ can be primitive itself is in diagonal action for $m = 2$. In this case it is not hard to find an element in T_i (just take a maximal nontrivial power of a random element try the normal closure).

Otherwise we can use further reduction to imprimitivity/intransitivity to find the socle components.

II.6.3. Chief Series. Computing a chief series is not much harder. The only difference is that we always have to ensure normality in the whole group. We can do this by simply intersecting conjugates.

LEMMA 86: Suppose that $N \lhd G$ and $M \lhd N$ with N/M simple. Then $L := \bigcap_{g \in G} M^g \lhd G$. If N/M is non-abelian then N/L is a minimal normal subgroup of G/L.

Proof: Homework □

If N/M (and thus N/L) is (elementary) abelian, we use Meataxe-type methods to reduce to chief factors.

In practice one would first compute the radical $R := O_\infty(G)$. Since this was obtained from iterated computations of p-cores $O_p(G)$ we have in fact already some splitup of R into chief factors and finish using the Meataxe.

The radical factor G/R then is treated using reduction to orbits, block systems etc.

II.7. Other groups with a natural action

There is a variety of other groups, which have naturally a faithful permutation action and thus could be treated like permutation groups. For example:

- Matrix groups $G \leq GL_n(p)$. Here the action is on vectors in $\mathbb{F}_p^n$.
- Groups of group automorphisms $G \leq \mathrm{Aut}(H)$. The action is on elements of H.

NOTE 87: We could (using the orbit algorithm) simply compute an isomorphic permutation group. However the permutation degree then tends to be large and for memory reasons it is often convenient to keep the original objects.

There is however one fundamental problem, for example for stabilizer chains: In general these groups have few short orbits. Thus, just picking random base points, will likely lead to very long orbits, often even to regular orbits (i.e. the first stabilizer is already trivial).

One can try to invest some work in finding short orbits (for matrix groups, for example base points that are eigenvector of random matrices or subgroups generated by random matrices have been proposed [MO95], as they guarantee the orbit to be not regular). In general, however this will not be sufficient.

Instead we consider *additional*, different, actions of G, which are related to the original action, but are not necessarily faithful. If H is the image of such an action, we would consider the permutation group $G \wr H$ with the whole factor group H glued together (so abstractly, the group is isomorphic G), acting intransitively. We then pick base points initially from the points moved by H, thus obtaining smaller orbit lengths. Once we need to pick base points from the original domain, we have a smaller group which automatically yields shorter orbits.

Since the second action can be obtained from the first, we do not really need to write down this pseudo-subdirect product, but simply consider different actions.

In terms of defining a stabilizer chain, each level of the chain simply carries a description of the appropriate action. Furthermore, we might switch the action multiple times in one stabilizer chain.

If G is a matrix group (over a field F of size > 2) an obvious related action is to act projectively, i.e. on 1-dimensional subspaces instead on vectors. This will typically reduce the initial orbit length by a factor of $|F - 1|$.

If G is a group of automorphisms of another group H, one can determine a characteristic (i.e. fixed under all automorphisms) subgroup $N \lhd H$ and initially consider the induced actions on N and on H/N.

Incidentally, it can be useful to do something similar for permutation groups: If the group is imprimitive, consider first the action on the blocks to get much shorter orbits.

the relator tables are:

a	a
1	1

b	b	b
1		1

a	$babababa$	b
1		1

and the subgroup tables are:

a	
1	1

b^{-1}	a	b
1		1

We now start defining new cosets by taking the image of an existing coset under a generator or its image. We enter the inverse information: If $a^g = b$ then $b^{g^{-1}} = a$. We also fill in all entries in the tables whose images become defined.

What may happen, is that such an entry fills the last hole in a table. Then we get an *deduction* that the image of one coset under the next generator must be the existing value on the other side of the table entry. We enter these deductions in the table as if they were definitions.

EXAMPLE 108 (Continued): In our example the first subgroup table immediately tells us that $1^a = 1$. We enter this in the coset table. (We will use underlines to denote definitions of cosets and bold numbers to denote the most recent change. An exclamation mark denotes a deduction.)

	a	a^{-1}	b	b^{-1}
1	**1**		**1**	

and update the other tables:

a	a
1	**1**!1

bbb	
1	1

a	$bababab$	
1	**1**	1

a	
1	1

$b^{-1}ab$	
1	1

We get a deduction $1^a = 1$, but this happens to be not new.

Because the subgroup tables carry only one row (for the trivial coset) we will retire the table for the generator a from now on.

Next we define coset 2 to be the image of coset 1 under b:

	a	a^{-1}	b	b^{-1}
1	**1**	1	**2**	
2				**1**

a	a
1	1
2	2

b	b	b
1	**2**	1
2	**1**	2

a	b	$ababab$	a	b
1	1	**2**		1
2			**1**	**1** 2

$b^{-1}ab$	
1	1

- The core (intersection of conjugates) of S is $(\varphi, \mathrm{Core}_G(U))$.
- If $S, T \leq G$ are both quotient subgroups given by the homomorphisms φ and ψ respectively, we can consider the larger quotient $\xi\colon G \to G^\varphi \perp G^\psi$ and calculate the intersection there.

If we have a quotient subgroup $S = \varphi^{-1}(U) \leq G$ the cosets $S\backslash G$ are in bijection to the cosets $U\backslash G^\varphi$. We can thus compare cosets or consider the action of G on the cosets of S. As S is the point stabilizer in this action, Schreier generators for this give a set of generators for S, thus converting a quotient subgroup in a "traditional" subgroup given by generators.

III.6. Coset Enumeration

In practice, we will often have subgroups given by generators and not as a quotient subgroup. Coset enumeration is a method that will produce the permutation representation on the cosets of this subgroup, provided it has finite index. This representation is an obvious choice to represent the subgroup as a quotient subgroup.

The fundamental idea behind the algorithm is that we perform an orbit algorithm on the cosets of the subgroup. As we do not have a proper element test we might not realize that certain cosets are the same, but we can eventually discover this using the defining relators of the group.

The method, one of the oldest group theoretic algorithms, was originally proposed for hand calculations. It is often named after the inventors as the "Todd-Coxeter algorithm" or simply as "Coset Enumeration".

NOTE 106: In view of theorem 100 the term "algorithm" is problematic (and ought to be replaced by "method"): The runtime cannot be bounded, that is the calculation may not finish in any preset finite time.

The main tool for coset enumeration is the *coset table*. It lists the cosets of the subgroup and for each coset the images under every generator and generator inverse. Coset 1 is defined to be the subgroup. Every other coset is defined to be the image of a prior coset under a generator.

We also maintain a table for every relator. These tables trace the images of every coset under the relator generator by generator. We know (as the relator has to be trivial in the group) that the coset needs to remain fixed.

Finally we keep a similar table for every subgroup generator, here however we only require the trivial coset to remain fixed.

EXAMPLE 107: Consider $G = \langle a, b \mid a^2 = b^3 = (ab)^5 = 1\rangle$ and $S = \langle a, a^b\rangle \leq G$. Then the coset table is

	a	a^{-1}	b	b^{-1}
1				

CHAPTER III

Finitely presented groups

Finitely presented groups are probably the most natural way to describe groups. Unfortunately they also are the computational least tractable and only afford a restricted set of methods.

In this chapter and the following we will often have to talk about generating systems, and about words (product expressions) in a particular generating system. If $\underline{\mathbf{g}}$ and $\underline{\mathbf{h}}$ are two sets of elements of the same cardinality, and $w(\underline{\mathbf{g}})$ is a product of elements of the one generating system, then we will write $w(\underline{\mathbf{h}})$ to mean the same product expression, but with every g_i replaced by h_i.

III.1. What are finitely presented groups

III.1.1. Free Groups.

DEFINITION 88: A group $F = \langle \underline{\mathbf{f}}\rangle$ is *free* on the generating set $\underline{\mathbf{f}}$ if every map $\underline{\mathbf{f}} \to H$ into a group H can be extended to a homomorphism $F \to H$.

NOTE 89: This is a property the basis of a vector space has.

It is not hard to show that the isomorphism type of a free group is determined by the cardinality of the generating system, we therefore will usually talk about a *free group of rank m*.

We now want to show that free groups exist. For this we consider an set of m letters: $x_1, x_2, \ldots, x_m$. (Or, if one prefers, $a, b, c, \ldots$.) We add m extra symbols $x_1^{-1}, \ldots, x_m^{-1}$, we call the resulting set of symbols our *alphabet* A.

For this alphabet A we consider the set A^* of *words* (i.e. sequences of letters, including the empty sequence) in A. Next we introduce an equivalence relation $\tilde{}$ on A^*: Two words in A are said to be directly equivalent, if one can be obtained from the other by inserting or deleting a sequence $x \cdot x^{-1}$ or $x^{-1}x$. We define $\tilde{}$ as the equivalence relation (smallest classes) on A^* induced by direct equivalence. We now consider $F = A^*/\tilde{}$. On this set we define the product of two classes as the class containing a concatenation of representatives. One then can show:

THEOREM 90: a) This product is well-defined.
b) F is a group
c) F is free on $x_1, \ldots, x_n$.

Proof: Tedious calculations: The hardest part is associativity as we have to consider multiple cases of cancellation. □

NOTE 91: By performing all cancellations, there is a shortest representative for every element of F, we will simply uses these representatives to denote elements of F.

III.1.2. Presentations. Now suppose that F is a free group of rank m. Then every group generated by m elements is isomorphic to a quotient F/N. We want to describe groups in such a way by giving a normal subgroup generating set for N.

DEFINITION 92: A *finitely presented group* G is a quotient $F/\langle R\rangle_F \cong G$ for a finite subset $R \subset F$. If $\underline{\mathbf{g}}$ is a free generating set for F we write $G \cong \left\langle \underline{\mathbf{g}} \mid R \right\rangle$ to describe this group and call this a *presentation* of G.
We call the elements of R a set of defining *relators* for G.

Instead of relators one sometimes considers *relations*, written in the form $l = r$. We will freely talk about relations with the interpretation that the corresponding relator l/r is meant.

NOTE 93: In general there will be many different presentations describing the same group.

NOTE 94: Besides being a convenient way for describing groups, finitely presented groups arise for example naturally in Topology, when describing the fundamental group of a topological space.

LEMMA 95: Every finite group is finitely presented

Proof: Suppose $|G| < \infty$. Choose a map $\varphi\colon F_{|G|} \to G$ that maps generators to the elements of G. It extends to a surjective homomorphism. Kern φ has finite index in $F_{|G|}$ and thus a finite number of Schreier generators. □

In this chapter we want to study algorithms for (finite or infinite) finitely presented groups.

NOTE 96: We will typically represent elements of a finitely presented group by their representatives in the free group, but we should be aware that


```
15:                fi;
16:              od;
17:            od;
18:            ...
19:          od;
20:        od;
21:  od;
22:  return L;
end
```

Note that this is not completely valid pseudo-code, as (lines 8 and 18) we permit a *variable* number of nested for-loops. In practice this has to be implemented recursively, or by using a while-loop that increments a list of variables.

NOTE 103: Note that the algorithm classifies homomorphisms, not quotient groups. If H has outer homomorphisms, we will get several homomorphisms with the same kernel.

NOTE 104: If we know that $|G| = |H|$ we will in fact find isomorphisms between G and H. In fact, if G and H are both permutation groups, once we determine a defining set of relators for G (section III.10) this approach offers a naive isomorphism test. In such a situation more restrictions on the generator images become available and help to reduce the search space.

If we set $G = H$ and run through all possibilities, we find automorphisms of G up to inner automorphisms and thus can determine generators for Aut(G).

There are newer and better algorithms for isomorphism and automorphism group of permutation groups.

III.5. Quotient subgroups

Staying with the homomorphism paradigm, the most convenient way to represent arbitrary subgroups of finite index is as pre-images under a homomorphism.

DEFINITION 105: Let G be a finitely presented group. A *quotient subgroup* (φ, U) of G is a subgroup $S \leq G$ that is given as preimage $S = \varphi^{-1}(U)$ of a subgroup $U \leq G^\varphi$ where $\varphi\colon G \to H$ is a homomorphism into a (typically finite) group.

The idea behind quotient subgroups is that we can calculate or test properties in the image, thus reducing for example to the case of permutation groups. For example:

- $g \in S$ if and only if $g^\varphi \in U = S^\varphi$.
- The quotient representation for $N_G(S)$ is $(\varphi, N_{G^\varphi}(U))$.

- The generator images generate H: $H = \left\langle \underline{\mathbf{g}}^{\varphi} \right\rangle$ (otherwise we just get a homomorphism.)

As $\underline{\mathbf{g}}$ and H are finite, there is just a finite set of generator images to consider for a given H, testing all therefore is a finite process.

If $\varphi\colon G \to H$ is an epimorphism and $h \in H$, the map $g \mapsto (g^{\varphi})^h$ is also an epimorphism, it is the product of φ and the inner automorphism of H induced by h. It therefore makes sense to enumerate images of the generators of G only up to inner automorphisms of H.

Suppose that $\underline{\mathbf{g}} = \{g_1, \ldots, g_m\}$ and the images are $\{h_1, \ldots, h_m\}$. If we permit conjugacy by h we can certainly achieve that h_1 is chosen to be a fixed representative in its conjugacy class. This reduces the possible conjugacy to elements of $C_1 = C_H(h_1)$.

Next h_2 can be chosen up to C_1 conjugacy. We can do this by first deciding on the H-class of h_2, say this class has representative r. Then the elements of r^H correspond to $C_H(r)\backslash H$. Thus C_1 orbits on this class correspond to the double cosets $C_H(r)\backslash H/C_1$. Conjugating r by representatives of these double cosets gives the possible candidates for h_2.

We then reduce conjugacy to $C_2 = C_H(h_1, h_2)$ and iterate on h_3.

This yields the following algorithm, called an GQuotient (here better: H-quotient) algorithm (Holt [HEO05] calls it EPIMORPHISMS):

ALGORITHM 102: Given a finitely presented group G and a finite group H, determine all epimorphisms from G to H up to inner automorphisms of H.

Input: $G = \langle g_1, \ldots, g_m \mid R \rangle$
Output: A list L of epimorphisms
begin
1: $L := []$;
2: Let C be a list of conjugacy class representatives for H
3: for $h_1 \in C$ do {Image of g_1}
4: for $r_2 \in C$ do {Class of image of g_2}
5: Let D_2 be a set of representatives of $C_H(r_2)\backslash H/C_H(h_1)$.
6: for $d_2 \in D_2$ do {Image of g_2}
7: $h_2 = r_2^{d_2}$;
8: ...
9: for $r_k \in C$ do {Class of image of g_k}
10: Let D_k be representatives of $C_H(r_k)\backslash H/C_H(h_1, h_2, \ldots, h_{k-1})$.
11: for $d_k \in D_k$ do {Image of g_k}
12: $h_k = r_k^{d_k}$;
13: if $\forall r \in R$: $r(h_1, \ldots, h_k) = 1$ and $H = \langle h_1, \ldots, h_k \rangle$ then
14: Add the map $g_i \mapsto h_i$ to L.

these representatives are not unique. Also there is in general no easy "normal form" as there is for small examples. (See chapter IV for more information about this.)

III.2. Tietze Transformations

There are some simple modifications of a presentation that do not change the group. They are called *Tietze Transformations*:

LEMMA 97: Suppose we have a presentation $G = \left\langle \underline{\mathbf{g}} \mid R \right\rangle$. Then the following transformations (called "Tietze Transformations") do not change G:

1. Add an extra relator that is a word in R.
2. Delete a relator that can be expressed as a word in the other relators.
3. For a word w in $\underline{\mathbf{g}}$, add a new generator x to $\underline{\mathbf{g}}$ and a new relator $x^{-1}w$ to R
4. If a generator $x \in \underline{\mathbf{g}}$ occurs only once and only in one relator, delete x and delete this relator.

Proof: Transformations 1 and 2 obviously do not change $\langle R \rangle_F$. For Transformations 3 and 4 there is an obvious map between the old and new groups, which preserves all relators and thus is an isomorphism. □

Tietze transformations were defined in the context of the following

LEMMA 98: Suppose that the presentations $P_1 = \left\langle \underline{\mathbf{g}} \mid R \right\rangle$ and $P_2 = \langle \underline{\mathbf{h}} \mid S \rangle$ yield isomorphic groups. Then there is a sequence of Tietze transformations from P_1 to P_2.

Proof: (Idea) If there is an isomorphism between P_1 and P_2 go first from P_1 to $Q = \left\langle \underline{\mathbf{g}} \cup \underline{\mathbf{h}} \mid R \cup S \cup T \right\rangle$ by adding relators T that express $\underline{\mathbf{h}}$ in terms of $\underline{\mathbf{g}}$ and deduce the relators in S, then delete the redundant $\underline{\mathbf{g}}$ by expressing them as words in $\underline{\mathbf{h}}$ to go from Q to P_2. □

This lemma itself is of little use, as the path of transformations between presentations is not known or known to be bounded in length.

They can however be used heuristically to try to simplify a presentation:

Only apply transformations which make a presentation immediately more simple; either by removing or shortening relators or by removing generators without increasing the overall length of the relators too much.

NOTE 99: By combining transformations, we get the following transformations which are more useful:

(1) Replace a relator r by $x^{-1}rx$ for $x \in \underline{\mathbf{g}}$. In particular, if $r = xa$ starts with x, this yields the cyclic permutation ax.
(2) If two relators overlap non-trivially: $r = abc$, $s = dbf$, we can use s to replace b in r: $r = ad^{-1}f^{-1}c$.
(3) If there is a relator in which one generator $x \in \underline{\mathbf{g}}$ occurs only once, say $r = axb$, then replace all occurrences of x by $a^{-1}b^{-1}$ and then delete x and r

In practice (such as the command `SimplifiedFpGroup` in GAP), Tietze transformations perform the following greedy algorithm by repeating the following steps:

(1) Eliminate redundant generators using relators of length 1 or 2 (this will not change the total relator length)
(2) Eliminate up to n (typically $n = 10$) generators, as long as the total relator length does not grow by more than m ($m = 30\%$)
(3) Find common substrings to reduce the total relator length, until the total improvement of a reduction round is less than p ($p = 0.01\%$).

Clearly this has no guarantee whatsoever to produce a "best" presentation, but at least often produces reasonable local minima.

III.3. Algorithms for finitely presented groups

The obvious aim for algorithms would be for example tests for finiteness or computation of group order, however there are some even more basic questions to be resolved first:

In what can be considered the first publication on computational group theory, in 1911 [Deh11] the mathematician Max Dehn asked for algorithms to solve the following problems (called "Dehn Problems" since then):

Word Problem: Given a finitely presented group G, is there an algorithm that decides whether a given word represents the identity in G?

Conjugacy Problem: Given a finitely presented group G, is there an algorithm that decides whether the elements represented by two words $u, v \in G$ are conjugate in G.

Isomorphism Problem: Is there an algorithm that decides whether a pair of finitely presented groups is isomorphic?

These problems have been resolved for some particular classes of presentations. In attacking any such questions in general, however, we are facing an unexpected obstacle, which shows that no such algorithms can exist:

THEOREM 100 (Boone [Boo57], Novikov [Nov55]): There cannot be an algorithm (in the Turing machine sense) that will test whether any given finitely presented group is trivial.

Proof: Translation to the Halteproblem (stopping problem) for Turing machines. □

Because of this problem the method we present may look strangely toothless, or may be only heuristics, but in fact this only reflect this fundamental problem.

III.4. Homomorphisms

There is *one* thing that is very easy to do with finitely presented groups, namely working with homomorphisms: We define homomorphisms by prescribing images of the generators. It is easy to test whether such a map is a homomorphism, as long as we can compare elements in the image group:

LEMMA 101: Let $G = \left\langle \underline{\mathbf{g}} \mid R \right\rangle$ be a finitely presented group. For a group H we define a map $\varphi\colon \underline{\mathbf{g}} \to H$. Then φ extends to a homomorphism $G \to H$ if and only if for every relator $r \in R$ we have that the relator evaluated in the generator images is trivial: $r(\underline{\mathbf{g}}^{\varphi}) = 1$.

Proof: Homework. □

Clearly evaluating such a homomorphism on an arbitrary element $w(\mathbf{g})$ simply means evaluating $w(\underline{\mathbf{g}}^{\varphi})$.

As this test is easy, much of the functionality for finitely presented groups involves homomorphisms – either working with homomorphic images, or finding homomorphisms (so-called "Quotient algorithms"). The easiest of these is probably to find epimorphisms onto a certain group:

III.4.1. Finding Epimorphisms. Given a finitely presented group $G = \left\langle \underline{\mathbf{g}} \mid R \right\rangle$ and another (finite) group H, we can find an epimorphism $\varphi\colon G \to H$ by trying to find suitable images $g_i^{\varphi} \in H$ for each generator $g_i \in \mathbf{g}$.

If we have a candidate set of images, they will yield an epimorphism if:

- The relators evaluated in the generator images are trivial: $r(\underline{\mathbf{g}}^{\varphi}) = 1_H$, and