30) a) Let G be a finite group, $N \lhd G$ and $F = G/N$. Let φ be the regular permutation representation of F. Show: G is isomorphic to a subgroup of $N \wr_\varphi F$. (i.e. F as a regular permutation group)
b) Let $N = A_6$ and $F = C_2$. We want to determine the isomorphism types of groups G that have a normal subgroup A_6 with a factor group C_2:

1. Construct (using `WreathProduct`) the wreath product $w := A_6 \wr C_2$.
2. Let `u:=List(ConjugacyClassesSubgroups(w),Representative)` be a list of subgroups of w (up to conjugacy).
3. Select (using `Filtered`) those subgroups from U that have order 720 and a derived subgroup of index 2.
4. You will end up with 5 subgroups. Calculate(using `NrConjugacyClasses`) the number of conjugacy classes for each. This shows that only two of these groups could be isomorphic. Use `IsomorphismGroups` to verify this.

(You will end up with 4 nonisomorphic groups. Which of them is S_6? What does this tell about the automorphism groups $\mathrm{Aut}(A_6)$ and $\mathrm{Aut}(S_6)$?)

31) If $G \le \mathrm{GL}_n(p)$ is a matrix group with $|G| > 1$, the following GAP command tests, whether G acts irreducibly on its natural module (i.e. if there are no invariant subspaces):

```
MTX.IsIrreducible(GModuleByMats(GeneratorsOfGroup(G),GF(p)));
```

a) Determine, using GAP, all subgroups of $\mathrm{GL}_3(2)$ (up to conjugacy) that act irreducibly.
Hint: `List(ConjugacyClassesSubgroups(GL(3,2)),Representative);` calculates a list of all subgroups up to conjugacy.
b) Construct the primitive groups of affine type of degree 8.

32) You are given the information that PSL(2,13), PGL(2,13), A_{14} and S_{14} are the only primitive groups of degree 14 and that A_{196} is the only simple group with a permutation representation of degree 196. Using the O'Nan-Scott Theorem and GAP determine the primitive groups of degree $14^2 = 196$.
Hint: First construct the possible socles. For each socle S calculate $N = N_{S_{196}}(S)$ and determine the subgroups $S \le U \le N$. These correspond to subgroups of N/S.

33) Let $G \le S_n$ be a transitive group and $\varphi \in \mathrm{Aut}(G)$. We say that φ is induced by $N_{S_n}(G)$ if there exists $h \in S_n$ such that $g^\varphi = g^h$ for every $g \in G$ (such an h must obviously normalize).
Show that φ is induced by S_n if and only if $\mathrm{Stab}_G(1)^\varphi = \mathrm{Stab}_G(j)$ for $1 \le j \le n$. (As the groups are finite, $\mathrm{Stab}_G(1)^\varphi \le \mathrm{Stab}_G(j)$ is sufficient.)
Hint: To show sufficiency, you will have to construct a suitable $h \in S_n$. Consider a bijection between the cosets $\mathrm{Stab}_G(1)\backslash G$ and $\mathrm{Stab}_G(1)^\varphi\backslash G$, induced by the automorphism φ. This bijection will yield the conjugating permutation.

34) Let G be a finite group. The radical $O_\infty(G)$ is the largest solvable normal subgroup of G. Let $F = G/O_\infty(G)$

a) Show that $\mathrm{Soc}(F)$ is the direct product of nonabelian simple groups.

b) Show that the action of F on $\mathrm{Soc}(F)$ is faithful, i.e. $F \leq \mathrm{Aut}(\mathrm{Soc}(F))$.

c) Show that F is a subdirect product of groups $A \leq \mathrm{Aut}(T) \wr S_m$, where T is a simple group and m the multiplicity of T in $\mathrm{Soc}(F)$.

35) Let $M \lhd G$ and $N \lhd M$ such that $M/N \cong T$ is simple nonabelian. We form $K = \mathrm{core}(N) = \bigcap_{g \in g} N^g$. Show that M/K is a minimal normal subgroup of G/K. (Hint: By going in the factor group G/K, you can assume WLOG that $K = \langle 1 \rangle$.)