30) a) Let $G$ be a finite group, $N \triangleleft G$ and $F = G/N$. Let $\varphi$ be the regular permutation representation of $F$. Show: $G$ is isomorphic to a subgroup of $N \wr \varphi F$. (i.e. $F$ as a regular permutation group)

b) Let $N = A_6$ and $F = C_2$. We want to determine the isomorphism types of groups $G$ that have a normal subgroup $A_6$ with a factor group $C_2$:

1. Construct (using WreathProduct) the wreath product $w := A_6 \wr C_2$.
2. Let $u := \text{List(ConjugacyClassesSubgroups}(w),\text{Representative})$ be a list of subgroups of $w$ (up to conjugacy).
3. Select (using Filtered) those subgroups from $U$ that have order 720 and a derived subgroup of index 2.
4. You will end up with 5 subgroups. Calculate (using NrConjugacyClasses) the number of conjugacy classes for each. This shows that only two of these groups could be isomorphic. Use IsomorphismGroups to verify this.

(You will end up with 4 nonisomorphic groups. Which of them is $S_6$? What does this tell about the automorphism groups $\text{Aut}(A_6)$ and $\text{Aut}(S_6)$?)

31) If $G \leq \text{GL}_n(p)$ is a matrix group with $|G| > 1$, the following GAP command tests, whether $G$ acts irreducibly on its natural module (i.e. if there are no invariant subspaces):

\[ \text{MTX.} \text{IsIrreducible}(\text{GModuleByMats}(\text{GeneratorsOfGroup}(G),\text{GF}(p))); \]

a) Determine, using GAP, all subgroups of $\text{GL}_3(2)$ (up to conjugacy) that act irreducibly.

**Hint:** $\text{List(ConjugacyClassesSubgroups}(\text{GL}(3,2)),\text{Representative});$ calculates a list of all subgroups up to conjugacy.

b) Construct the primitive groups of affine type of degree 8.

32) You are given the information that $\text{PSL}(2,13)$, $\text{PGL}(2,13)$, $A_{14}$ and $S_{14}$ are the only primitive groups of degree 14 and that $A_{196}$ is the only simple group with a permutation representation of degree 196. Using the O’Nan-Scott Theorem and GAP determine the primitive groups of degree $14^2 = 196$.

**Hint:** First construct the possible socles. For each socle $S$ calculate $N = N_{S_{196}}(S)$ and determine the subgroups $S \leq U \leq N$. These correspond to subgroups of $N/S$.

33) Let $G \leq S_n$ be a transitive group and $\varphi \in \text{Aut}(G)$. We say that $\varphi$ is induced by $N_{S_n}(G)$ if there exists $h \in S_n$ such that $g^\varphi = g^h$ for every $g \in G$ (such an $h$ must obviously normalize).

Show that $\varphi$ is induced by $S_n$ if and only if $\text{Stab}_G(1)^\varphi = \text{Stab}_G(j)$ for $1 \leq j \leq n$. (As the groups are finite, $\text{Stab}_G(1)^\varphi \leq \text{Stab}_G(j)$ is sufficient.)

**Hint:** To show sufficiency, you will have to construct a suitable $h \in S_n$. Consider a bijection between the cosets $\text{Stab}_G(1)^\varphi G$ and $\text{Stab}_G(1)^\varphi G$, induced by the automorphism $\varphi$. This bijection will yield the conjugating permutation.
34) Let $G$ be a finite group. The radical $O_\infty(G)$ is the largest solvable normal subgroup of $G$. Let $F = G/O_\infty(G)$

a) Show that $\text{Soc}(F)$ is the direct product of nonabelian simple groups.

b) Show that the action of $F$ on $\text{Soc}(F)$ is faithful, i.e. $F \leq \text{Aut}(\text{Soc}(F))$.

c) Show that $F$ is a subdirect product of groups $A \leq \text{Aut}(T) \wr S_m$, where $T$ is a simple group and $m$ the multiplicity of $T$ in $\text{Soc}(F)$.

35) Let $M \triangleleft G$ and $N \triangleleft M$ such that $M/N \cong T$ is simple nonabelian. We form $K = \text{core}(N) = \bigcap_{g \in G} N^g$. Show that $M/K$ is a minimal normal subgroup of $G/K$. (Hint: By going in the factor group $G/K$, you can assume WLOG that $K = \langle 1 \rangle$.)