Mathematics 676 **Homework (due Apr 17)** A. Hulpke

27) [High nutritious value]
Let M be the monoid with the presentation $\langle x, y \mid x^3 = 1, y^3 = 1, (xy)^3 = 1\rangle$.
a) Determine a confluent rewriting system for M with respect to length+lexicographic ordering.
b) Using the rules determined in a), construct an infinite set of words in normal form, thus proving that M is infinite.

28) If you define a finitely presented group in GAP, you can enforce reduction of elements to normal form (running a length/lex Knuth-Bendix algorithm on the presentation to obtain a confluent rewriting system) by calling `SetReducedMultiplication(G);` Why do you think this is not turned on by default?

29) Give an example of a finite monoid that is not a group.

30) [High nutritious value]
Let $G = \langle a = (1,2,3)(4,5,6), b = (3,4)\rangle$, and $N = \langle (2,6)(3,4), (1,5)(3,4)\rangle \lhd G$. Then $|G| = 24$, $N \cong C_2^2$ and $G/N \cong C_6 \cong C_2 \times C_2$, gerated by the element $c = Na$ of order 3 and $d = Nb$ of order 2. Thus

$$\left\langle c, d \mid c^3 \to 1, d^2 \to 1, dc \to cd\right\rangle$$

is a confluent rewriting system for G/N. Furthermore, $p^a = q$, $q^a = pq$, $p^b = p$, $q^b = q$. Using the method described in the lecture, determine a linear system of equations that describes the complements of N in G. (You may use GAP for calculations.)

31) [bonus]
The plactic monoid on n generators is defined by the rules $yzx = yxz$ whenever $x < y \le z$ and $xzy = zxy$ whenever $x \le y < z$.
a) Construct a confluent rewriting system for the plactic monoid on 3 generators.
b) Show that the plactic monoid on 4 or more generators does not have a finite confluent rewriting system (with respect to the length+lex ordering).

32) [bonus]
Let G be a group and $M \lhd G$ an elementary abelian normal subgroup. We choose a set of representatives for $F := G/M$, let $\tau: F \to G$ be this representative map. We call

$$Z^1(F, M) := \left\{\gamma: F \to M \mid (fg)^\gamma = (f^\gamma)^{g^\tau} g^\gamma \forall f, g \in F\right\}$$

the group of 1-cocyles and

$$B^1(F, M) := \left\{\gamma_m = (f \mapsto m^{-f^\tau} m): F \to M \mid m \in M\right\}$$

the group of 1-coboundaries. Show:
a) Z^1 is a group (under pointwise multiplication of functions) and $B^1 \le Z^1$. We call $H^1 = Z^1/B^1$ the 1-cohomology group.

b) Suppose that $A\underline{\mathbf{x}} = \underline{\mathbf{b}}$ is the system of linear equations used to determine complements to M in G. Show that Z^1 corresponds to the solutions of the associated homogeneous system $A\underline{\mathbf{x}} = \underline{\mathbf{0}}$.
c) Assuming that there is a complement C to M in G and that the representative map $\tau\colon F \to C$ is in fact an isomorphism (in this situation the system of equations to determine complements is homogeneous), show that there is a bijection between Z^1 and the set of complements to M in G. d) Show that two complements are conjugate under G if and only if they are conjugate under M, and that this is the case if and only if the corresponding cocycles γ, δ (using the bijection found in c) fulfill that γ and δ are in the same coset of B^1.