17) a) For a prime p, consider $G \leq S_p$ generated by two p-cycles. Show that G must be simple.
b) (Bonus) How many nonisomorphic groups can arise in this situation for $p = 5$, $p = 7$ or $p = 11$?

18) a) Let G be a finite group, $N \lhd G$ and $F = G/N$. Let φ be the regular permutation representation of F. Show: G is isomorphic to a subgroup of $N \wr_\varphi F$. (i.e. F as a regular permutation group)
b) Let $N = A_6$ and $F = C_2$. We want to determine, using GAP the isomorphism types of groups G that have a normal subgroup A_6 with a factor group C_2:

1. Construct (using `WreathProduct`) the wreath product $W := A_6 \wr C_2$.
2. Let `u:=List(ConjugacyClassesSubgroups(W),Representative)` be a list of subgroups of w (up to conjugacy).
3. Select (using `Filtered`) those subgroups from U that have order 720 and a derived subgroup of index 2.
4. You will end up with 5 subgroups. Calculate (using `NrConjugacyClasses`) the number of conjugacy classes for each. This shows that only two of these groups could be isomorphic. Use `IsomorphismGroups` to verify this.

(You will end up with 4 nonisomorphic groups. Which of them is S_6? What does this tell about the automorphism groups $\mathrm{Aut}(A_6)$ and $\mathrm{Aut}(S_6)$?)

19) [High nutritious value] Let $G \leq S_n$ be a transitive group and $\varphi \in \mathrm{Aut}(G)$. We say that φ is *realized by* S_n, if there exists $h \in N_{S_n}(G)$ such that $g^\varphi = g^h = h^{-1}gh$ for every $g \in G$ (such an h must obviously normalize).
Show that φ is realized by S_n if and only if $\mathrm{Stab}_G(1)^\varphi = \mathrm{Stab}_G(j)$ for $1 \leq j \leq n$. (As the groups are finite, $\mathrm{Stab}_G(1)^\varphi \leq \mathrm{Stab}_G(j)$ is sufficient.)
Hint: To show sufficiency, you will have to construct a suitable $h \in S_n$. Consider a bijection between the cosets of $\mathrm{Stab}_G(1)$ in G and those of $\mathrm{Stab}_G(1)^\varphi$ in G, induced by the automorphism φ. This bijection will yield the conjugating permutation.

20) [High nutritious value] Let G be a finite group. The *radical* $O_\infty(G)$ is the largest solvable normal subgroup of G. Let $F = G/O_\infty(G)$
a) Show that $\mathrm{Soc}(F)$ is the direct product of nonabelian simple groups.
b) Show that the action of F on $\mathrm{Soc}(F)$ is faithful, i.e. $F \leq \mathrm{Aut}(\mathrm{Soc}(F))$.
c) Show that F is a subdirect product of groups $A \leq \mathrm{Aut}(T) \wr S_m$, where T is a simple group and m the multiplicity of T in $\mathrm{Soc}(F)$.

21) Show that $G = \langle x, y \mid x^2, y^2 \rangle$ is infinite.
Hint: Find a group (e.g. a subgroup of $\mathrm{GL}(2, \mathbb{Q})$) that must be a quotient of G, but contains elements of infinite order.