58) Let $G = \langle (1, 3)(2,8)(4,6)(5,7), (1, 6)(2,7,3,5,8,4) \rangle$ and $N = \langle \rangle \triangleleft G$. Then $|G| = 24$ and $N \cong C_2^2$. Using the method described in the lecture, determine a linear system of equations that describes the complements of $N$ in $G$. (You may use GAP for calculations, such as obtaining a presentation for $G/N$.)

59) Let $G$ be a group and $M \triangleleft G$ an elementary abelian normal subgroup. We choose a set of representatives for $F := G/M$, let $\tau: F \to G$ be this representative map. We call

$$Z^1(F, M) := \{ \gamma: F \to M \mid (f g)\gamma = (f \gamma)g \forall f, g \in F \}$$

the group of 1-cocycles and

$$B^1(F, M) := \{ \gamma_m = (f \mapsto m^{-f} m): F \to M \mid m \in M \}$$

the group of 1-coboundaries. Show:

a) $Z^1$ is a group (under pointwise multiplication of functions) and $B^1 \leq Z^1$. We call $H^1 = Z^1/B^1$ the 1-cohomology group.

b) Suppose that $A \mathbf{x} = \mathbf{b}$ is the system of linear equations used to determine complements to $M$ in $G$. Show that $Z^1$ corresponds to the solutions of the associated homogeneous system $A \mathbf{x} = \mathbf{0}$.

c) Assuming that there is a complement $C$ to $M$ in $G$ and that the representative map $\tau: F \to C$ is in fact an isomorphism (in this situation the system of equations to determine complements is homogeneous), show that there is a bijection between $Z^1$ and the set of complements to $M$ in $G$. d) Show that two complements are conjugate under $G$ if and only if they are conjugate under $M$ if and only if the corresponding cocycles (using the bijection found in c) $\gamma, \delta$ fulfill that $\gamma$ and $\delta$ are in the same coset of $B^1$.

60) In this problem, we want to construct all groups $H$ of order 16, such that $|H/\Phi(H)| = 2^2$. Consider groups of the form $\langle a, b \mid a^k = b^l = 1 \rangle$ with $k, l \in \{2, 4, 8\}$, and use the $p$-Quotient algorithm in GAP (EpimorphismPGroup) to determine quotients (of class up to 3). The desired groups must be quotients of the images obtained. (There are up to isomorphism 2 abelian, and 6 nonabelian groups of this kind.)

61) a) Let $G$ be a group and $S_1, \ldots, S_k \leq G$. Show: The (simultaneous, intransitive) action of $G$ on the cosets of the $S_i$ is faithful if and only if $\cap_i \text{core}_G(S_i) = \{1\}$.

b) Using the characterization in a), write a program in GAP which determines for a group $G$ the smallest degree of a faithful permutation representation. (You may use GAP, in particular the function ConjugacyClassesSubgroups to determine the subgroups of $G$.)

c) Using the library of small groups in GAP, find an example of a group $G$ and $N \triangleleft G$ such that the smallest degree of a faithful permutation representation of $G$ is smaller than that of $G/N$. 
Let $p$ be a prime. Describe an algorithm – using the theory of normal forms – to write down representatives of the conjugacy classes of $\text{GL}_n(p)$. You may assume that (by factoring $x^{(p^n)} - x$) you have a method that can produce all irreducible polynomials of a given degree over $GF(p)$. 