58) Let $G = \langle (1,3)(2,8)(4,6)(5,7), (1,6)(2,7,3,5,8,4) \rangle$ and $N = \langle \rangle \lhd G$. Then $|G| = 24$ and $N \cong C_2^2$. Using the method described in the lecture, determine a linear system of equations that describes the complements of N in G. (You may use GAP for calculations, such as obtaining a presentation for G/N.)

59) Let G be a group and $M \lhd G$ an elementary abelian normal subgroup. We choose a set of representatives for $F := G/M$, let $\tau: F \to G$ be this representative map. We call

$$Z^1(F,M) := \left\{ \gamma: F \to M \mid (fg)^\gamma = (f^\gamma)^{g^\tau} g^\gamma \forall f, g \in F \right\}$$

the group of 1-cocyles and

$$B^1(F,M) := \left\{ \gamma_m = (f \mapsto m^{-f^\tau} m): F \to M \mid m \in M \right\}$$

the group of 1-coboundaries. Show:
a) Z^1 is a group (under pointwise multiplication of functions) and $B^1 \le Z^1$. We call $H^1 = Z^1/B^1$ the 1-cohomology group.
b) Suppose that $A\underline{\mathbf{x}} = \underline{\mathbf{b}}$ is the system of linear equations used to determine complements to M in G. Show that Z^1 corresponds to the solutions of the associated homogeneous system $A\underline{\mathbf{x}} = \underline{\mathbf{0}}$.
c) Assuming that there is a complement C to M in G and that the representative map $\tau: F \to C$ is in fact an isomorphism (in this situation the system of equations to determine complements is homogeneous), show that there is a bijection between Z^1 and the set of complements to M in G. d) Show that two complements are conjugate under G if and only if they are conjugate under M if and only if the corresponding cocycles (using the bijection found in c) γ, δ fulfill that γ and δ are in the same coset of B^1.

60) In this problem, we want to construct all groups H of order 16, such that $|H/\Phi(H)| = 2^2$. Consider groups of the form $\langle a, b \mid a^k = b^l = 1 \rangle$ with $k, l \in \{2,4,8\}$, and use the p-Quotient algorithm in GAP (`EpimorphismPGroup`) to determine quotients (of class up to 3). The desired groups must be quotients of the images obtained. (There are up to isomorphism 2 abelian, and 6 nonabelian groups of this kind.)

61) a) Let G be a group and $S_1, \ldots, S_k \le G$. Show: The (simultaneous, intransitive) action of G on the cosets of the S_i is faithful if and only if $\bigcap_i \mathrm{core}_G(S_i) = \langle 1 \rangle$.
b) Using the characterization in a), write a program in GAP which determines for a group G the smallest degree of a faithful permutation representation. (You may use GAP, in particular the function `ConjugacyClassesSubgroups` to determine the subgroups of G.)
c) Using the library of small groups in GAP, find an example of a group G and $N \lhd G$ such that the smallest degree of a faithful permutation representation of G is smaller than that of G/N.

62) Let p be a prime. Describe an algorithm – using the theory of normal forms – to write down representatives of the conjugacy classes of $\mathrm{GL}_n(p)$. You may assume that (by factoring $x^{(p^a)} - x$) you have a method that can produce all irreducble polynomials of a given degree over $GF(p)$.