37) Let \( G \triangleleft A \) and \( \phi : G \to \text{GL}(V) \) a representation.
   a) Let \( a \in A \). We define \( \phi^a : g \to (a^{-1}ga)^\phi \). Show that \( \phi^a \) is a representation of \( G \) which is irreducible if and only if \( \phi \) is irreducible.
   b) Now consider that \( [A : G] = 2 \). Show that for a given irreducible representation \( \phi : G \to \text{GL}(V) \) either:
      i) \( \phi^a = \phi \forall a \in A \).
      ii) There exists \( b \in A \) such that \( \phi \neq \phi^b \) and for any \( a \in A \) we have that \( \phi^a \in \{ \phi, \phi^b \} \). (I.e. there exactly two conjugates of \( \phi \).)

38) Show (by a combinatorial argument) that for \( G = S_n \) the classes \((1,n)^G, (1,2,3,\ldots,n)^G \) and \((2,3,\ldots,n)^G \) form a rational rigid triple and that \( G = \langle (1,n), (1,2,3,\ldots,n) \rangle \). (Thus \( G \) occurs as Galois group over \( \mathbb{Q} \).)

39) Let \( G \) be a finite group. Define
   \[ \Phi(G) = \bigcap_{M_{\max G} \triangleleft G} \Phi \]
   to be the intersection of all maximal subgroups of \( G \). \( \Phi(G) \) is called the Frattini-subgroup of \( G \).
   a) Show that
   \[ \Phi(G) = \{ g \in G \mid \forall X \subset G : (\text{if } G = \langle X \cup \{ g \} \rangle \text{ then } G = \langle X \rangle) \} , \]
   that is \( \Phi(G) \) consists of those elements of \( G \) that are redundant in every generating set.
   b) Show that for any subset \( X = \{ x_1, m_1, \ldots, x_m \} \subset G \) that
   \[ G = \langle X \rangle \iff G/\Phi(G) = \langle \Phi(G)x_1, \ldots, \Phi(G)x_m \rangle \]

40) Let \( G \) be a group with \( |G| = p^a \) for \( p \) prime.
   a) Show that if \( U < G \) then \( U \subseteq N_G(U) \). (Hint: Consider the action of \( U \) on the cosets of \( U \) in \( G \) and show that the coset \( Ug \) is a fixed point (i.e. \( Ugu = Ug \forall u \in U \)) if and only if \( g \in N_G(U) \).)
   b) Show that any maximal subgroup of \( G \) has index \( p \) and is normal in \( G \).
   c) Let \( N = G'G^{p^a} = \langle \{ x, y \}^G, g^p \mid x, y, g \in G \rangle \). Show that \( N = \Phi(G) \) and that \( G/N \cong C_p \times \cdots \times C_p \) is a direct product of cyclic \( p \)-groups (i.e. an \( \mathbb{F}_p \) vector space).
   Note: Problem 39b) now implies that \( \dim(G/N) \) is the exact cardinality of an irredundant generating set for \( G \), and a subset of this cardinality generates \( G \) if and only if it generates \( G/N \). This is called Burnside's basis theorem.

41') The benzene molecule is a planar ring of 6 carbon molecules with attached hydrogen molecules. (The carbon molecules have 3 double bonds in common, but these are not localized. For purposes of symmetry we may assume that a 60 degree rotation is possible.)
   a) What is the symmetry group of benzene?
   b) Suppose that all vibrations of benzene take place in the plane. Also assume that the hydrogen molecules are tightly bound to the carbon and thus consider the hydrogen/carbon pair as one unit. The vibrations of benzene are thus described by \( \mathbb{R}^{12} \).
   Choose coordinates and calculate the character for the action of the symmetry group on \( \mathbb{R}^{12} \).
   c) Express \( \mathbb{R}^{12} \) as a direct sum of homogeneous components and deduce a set of normal modes for benzene.

Problems marked with * are bonus problems for extra credit.