29) Let K be algebraically closed and $f, g \in K[x, y]$. Show that $\mathcal{V}((f, g))$ is infinite if and only if f, g have a nonconstant common factor.

30) Let R be an integral domain with quotient field F. Let $\bar{F}$ be the algebraic closure of F. We then can write polynomials over R as product of linear factors over $\bar{F}$.
Let $f, g \in R[x]$ such that $f(x) = a \cdot \prod_{i=1}^{m}(x - \alpha_i)$ and $g(x) = b \cdot \prod_{j=1}^{n}(x - \beta_j)$ in $\bar{F}[x]$.
a) Show that the resultant (with respect to x) satisfies:

$$\mathrm{Res}_x(f, g) = a^n b^m \prod_{i,j}(\alpha_i - \beta_j) = (-1)^{mn} b^m \prod_j f(\beta_j) = a^n \prod_i g(\alpha_i)$$

b) Show that (embedding $R[x]$ into $R[x, y]$)

i) $\mathrm{Res}_y(f(x - y), g(y))$ has roots $\alpha_i + \beta_j$.

ii) $\mathrm{Res}_y(y^m f(x/y), g(y))$ has roots $\alpha_i \cdot \beta_j$. (Note that $y^m f(x/y) \in R[x, y]$.)

31) a) Let $\alpha = \sqrt{2} + \sqrt{3}$. Calculate a polynomial $f \in \mathbb{Z}[x]$ of degree 4, such that $f(\alpha) = 0$.
b) Show that $\sqrt{2}, \sqrt{3} \in \mathbb{Q}(\alpha)$, and conclude that the polynomial f from a) must be irreducible. (For this, you do not need to know the concrete form of the polynomial.) Show that $\mathbb{Q}(\alpha) > \mathbb{Q}$ is a Galois extension with Galois group $C_2 \times C_2$. Deterine the subfields of $\mathbb{Q}(\alpha)$.
c) Let $p \geq 5$ be a prime. Show that if neither 2, nor 3, is a square modulo p then 6 must be a square modulo p. Conclude that the polynomial f from a) is reducible modulo every prime (but is irreducible over $\mathbb{Z}$).

32) Let $A \in K^{a \times a}$, $B \in K^{b \times b}$ be two matrices (of not necessarily the same dimension) and $M = A \otimes B$ their Kronecker product.
a) Assuming that the characteristic polynomial of A is $m_A = \prod_{i=1}^{a}(x - \alpha_i)$ and the one of B is $m_B = \prod_{i=1}^{b}(x - \beta)$, what is the characteristic polynomial of M?
b) Find, for the case $a = b = 2$, a nontrivial relation that holds amongst the coefficients of the minimal polynomial of M. (Express, with a computer, the coefficients of the characteristic polynomial of M in terms of the α_i, β_j and find a syzygy condition amongst them.)