25) Calculate a Gröbner basis for

\[ I = (x^2y - 1, xy^2 - 1) \]

using the lex ordering.

From now on you may use a computer algebra system to calculate Gröbner bases.

26) Let \( I = \langle xy^3 - x^2, x^3y^2 - y \rangle \triangleleft \mathbb{Q}[x, y] = R. \)

a) Compute a (nonreduced to avoid messy coefficients) Gröbner basis for \( I \) with respect to \textit{deglex} ordering. Determine the possible form of canonical representatives for cosets in \( R/I \). What is the dimension of \( R/I \) as a \( \mathbb{Q} \) vector space.

b) The maps \( \alpha: R/I \to R/I, I + p \mapsto I + x \cdot p \) and \( \beta: R/I \to R/I, I + p \mapsto I + y \cdot p \) are \( \mathbb{Q} \)-vector-space homomorphisms of \( R/I \). (Persuade yourself that they are, but you do not need to show this.) Compute matrices \( M_\alpha \) and \( M_\beta \) for \( \alpha \), respectively \( \beta \) (with respect to the basis found in part a).

c) Show that the map \( \varphi: R \to \mathbb{Q}^{d \times d} \) (where \( d \) is the appropriate dimension), \( f(x, y) \mapsto f(M_\alpha, M_\beta) \) is a ring homomorphism with kernel \( I \). (In other words: we can compute in \( R/I \) by computing with these matrices instead.)

27) Consider the parametric curve

\[ x = \frac{1 - t^2}{1 + t^2}, \quad y = \frac{2t}{1 + t^2}. \]

Describe this curve by polynomials in \( x, y, \) and \( t \). By eliminating \( t \), determine a polynomial in \( x \) and \( y \) describing the curve and use this result to identify the curve.

28) Let \( R = \mathbb{Q}[x, y, z] \) and \( I = \langle x^2 + yz - 2, y^2 + xz - 3, xy + z^2 - 5 \rangle \triangleleft R. \) Show that \( x + I \) is a unit in \( R/I \) and determine \( (x + I)^{-1}. \)

29) (Ideal Intersection and Multivariate Gcd)

a) Let \( F \) be a field and \( R = F[x_1, \ldots, x_n] \) and let \( I_1 = \langle f_1, \ldots, f_k \rangle \triangleleft R \) and \( I_2 = \langle h_1, \ldots, h_r \rangle \triangleleft R \) be two ideals. Let \( S = F[x_1, \ldots, x_n, t] \) (considering \( R \subset S \)) and set

\[ J = \langle t \cdot f_1, \ldots, t \cdot f_k, (1 - t) \cdot h_1, \ldots, (1 - t) \cdot h_r \rangle \triangleleft S. \]

Show that \( I_1 \cap I_2 = J \cap R. \)

b) Let \( f = x^3 z^2 + x^2 y z^2 - x y^2 z^2 - y^3 z^2 + x^4 + x^3 y - x^2 y^2 - x y^3 \) and \( g = x^2 z^4 - y^2 z^4 + 2x^3 z^2 - 2x y^2 z^2 + x^4 - x^2 y^2. \) Compute \( \langle f \rangle \cap \langle g \rangle. \)

c) Compute \( \gcd(f, g). \) (Hint: Show that \( \langle f \rangle \cap \langle g \rangle = \langle \text{lcm}(f, g) \rangle. \))
30) (Intended to illustrate the reason for having different monomial orderings.) Let \( I = \langle x^5 + y^4 + z^3 - 1, x^3 + y^2 + z^2 - 1 \rangle \). Compute a Gröbner basis for \( I \) with respect to the \textit{lex} and \textit{deglex} orderings. Compare. Repeat the calculations for \( I = \langle x^5 + y^4 + z^3 - 1, x^3 + y^3 + z^2 - 1 \rangle \) (only one exponent changed!)

31) Consider the curve, described in polar coordinates by the equation \( r = 1 + \cos(2\theta) \). We want to describe this curve by an equation in \( x \) and \( y \):

a) Write equations for the \( x \)- and \( y \)-coordinates of points on this curve parameterized by \( \theta \).

b) Take the equations of a) and write them as polynomials in the new variables \( \sin \theta = z \) and \( \cos \theta = w \).

c) Using the fact that \( z^2 + w^2 = 1 \), determine a single polynomial in \( x \) and \( y \) whose zeroes are the given curve. Does this polynomial have factors (GAP command \texttt{Factors})?

32) A theorem of Euclid (book IV, proposition 5) states, that for a triangle \( ABC \) the lines bisecting the sides perpendicularly intersect in the center of the outer circle (the circle through the vertices, also called \textit{circumcircle}) of the triangle. Prove this theorem using coordinates and polynomials.

\textbf{Hint}: For reasons of symmetry, it is sufficient to assume that \textit{two} of the perpendicular lines intersect in this point, or that the center lies on each perpendicular line. You may also assume by scaling and rotating that \( A = (0, 0) \) and \( B = (0, 1) \).

33) Compute the kernel of the ring homomorphism \( \mathbb{Q}[x, y, z] \to \mathbb{Q}[t]/(t^{12}) \) defined by

\[ x \mapsto t^5, \quad y \mapsto t^7 + t^8, \quad z \mapsto t^{11}. \]