11) Calculate a Gröbner basis for
\[ I = (x^2y - 1, xy^2 - 1) \]
using the lex ordering.

From now on you may use a computer algebra system to calculate Gröbner bases.

12) Let \( I = (xy^3 - x^2, x^3y^2 - y) \triangleleft \mathbb{Q}[x, y] = R. \)
a) Compute a (nonreduced to avoid messy coefficients) Gröbner basis for \( I \) with respect to deglex ordering. Determine the possible form of canonical representatives for cosets in \( R/I \). What is the dimension of \( R/I \) as a \( \mathbb{Q} \) vector space.
b) The maps \( \alpha: R/I \to R/I, I + p \mapsto I + x \cdot p \) and \( \beta: R/I \to R/I, I + p \mapsto I + y \cdot p \) are \( \mathbb{Q} \)-vector-space homomorphisms of \( R/I \). (Persuade yourself that they are, but you do not need to show this.) Compute matrices \( M_\alpha \) and \( M_\beta \) for \( \alpha \), respectively \( \beta \) (with respect to the basis found in part a)).
c) Show that the map \( \varphi: R \to \mathbb{Q}^{d\times d} \) (where \( d \) is the appropriate dimension), \( f(x, y) \mapsto f(M_\alpha, M_\beta) \) is a ring homomorphism with kernel \( I \). (In other words: we can compute in \( R/I \) by computing with these matrices instead.)

13) Consider the parametric curve
\[ x = \frac{1 - t^2}{1 + t^2}, \quad y = \frac{2t}{1 + t^2}. \]
Describe this curve by polynomials in \( x, y, \) and \( t \). By eliminating \( t \), determine a polynomial in \( x \) and \( y \) describing the curve and use this result to identify the curve.

14) Let \( R = \mathbb{Q}[x, y, z] \) and \( I = (x^2 + yz - 2, y^2 + xz - 3, xy + z^2 - 5) \triangleleft R. \) Show that \( x + I \) is a unit in \( R/I \) and determine \( (x + I)^{-1} \).

15) (Ideal Intersection and Multivariate Gcd)
a) Let \( F \) be a field and \( R = F[x_1, \ldots, x_n] \) and let \( I_1 = (f_1, \ldots, f_k) \triangleleft R \) and \( I_2 = (h_1, \ldots, h_r) \triangleleft R \) be two ideals. Let \( S = F[x_1, \ldots, x_n, t] \) (considering \( R \subset S \)) and set
\[ J = (t \cdot f_1, \ldots, t \cdot f_k, (1 - t) \cdot h_1, \ldots, (1 - t) \cdot h_r) \triangleleft S. \]
Show that \( I_1 \cap I_2 = J \cap R. \)
b) Let \( f = x^3z^2 + x^2y^2z^2 - x^2y^2z^2 - y^3z^2 + x^4 + x^3y - x^2y^2 - xy^3 \) and \( g = x^2z^4 - y^2z^4 + 2x^3z^2 - 2xy^2z^2 + x^4 - x^2y^2 \). Compute \( (f) \cap (g) \).
c) Compute \( \gcd(f, g) \). (Hint: Show that \( (f) \cap (g) = (\lcm(f, g)) \).)
16) (Intended to illustrate the reason for having different monomial orderings.) Let $I = (x^2 + y^4 + z^3 - 1, x^3 + y^3 + z^2 - 1)$. Compute a Gröbner basis for $I$ with respect to the lex and deglex orderings. Compare. Repeat the calculations for $I = (x^5 + y^4 + z^3 - 1, x^3 + y^3 + z^2 - 1)$ (only one exponent changed!)

17) Consider the curve, described in polar coordinates by the equation $r = 1 + \cos(2\theta)$. We want to describe this curve by an equation in $x$ and $y$:
   a) Write equations for the $x$- and $y$-coordinates of points on this curve parameterized by $\theta$.
   b) Take the equations of a) and write them as polynomials in the new variables $\sin \theta = z$ and $\cos \theta = w$.
   c) Using the fact that $z^2 + w^2 = 1$, determine a single polynomial in $x$ and $y$ whose zeroes are the given curve. Does this polynomial have factors (GAP command Factors)?

18) A theorem of Euclid (book IV, proposition 5) states, that for a triangle $ABC$ the lines bisecting the sides perpendicularly intersect in the center of the outer circle (the circle through the vertices, also called circumcircle) of the triangle. Prove this theorem using coordinates and polynomials. Hint: For reasons of symmetry, it is sufficient to assume that two of the perpendicular lines intersect in this point, or that the center lies on each perpendicular line. You may also assume by scaling and rotating that $A = (0, 0)$ and $B = (0, 1)$.

19) Compute the kernel of the ring homomorphism $\mathbb{Q}[x, y, z] \rightarrow \mathbb{Q}[t]/(t^{12})$ defined by $x \mapsto t^5, y \mapsto t^7 + t^8, z \mapsto t^{11}$. 