26) a) Let $\zeta_1, \dots \zeta_k \in \overline{\mathbb{Q}}$ be roots of unity (not necessarily different). Show that if $\sum_{i=1}^k \zeta_i = k$ then $\zeta_i = 1$ for all i. Show that if (absiolute value) $\left|\sum_{i=1}^k \zeta_i\right| = k$, then $\zeta_i = \zeta_j$ for all i, j.
b) Let G be a finite group and χ be a character, associated to the representation $\varphi\colon G \to \mathrm{GL}_n(\mathbb{C})$. We define the *kernel of* χ as the subset $\ker\chi = \{g \in G \mid \chi(g) = \chi(1_G)\} \subset G$. Show that $\ker\chi = \ker\varphi$.
c) We define the *center of a character* χ as $Z(\chi) = \{g \in G \mid \chi(1) = |chi(g)|\}$. Show that $Z(\chi)$ is the set of $g \in G$ such that $\varphi(g) \in Z(\varphi(G))$.
d) Show that if $\chi = \psi + \rho$ for two characters ψ and ρ, then $\ker\chi = \ker\psi \cap \ker\rho$. Conclude that the character table of G determines the normal subgroups of G (consider the action on the cosets of a normal subgroup).
e) Show (using the regular representation) that

$$Z(G) = \bigcap_{\chi \in \mathrm{Irr}(G)} Z(\chi),$$

thus the character table also determines the centre of G.

27) Show that the sum over a column of a character table is an integer. (Note: it can be negative.)

28) Let G be a finite group with conjugacy classes $K_1, \dots, K_n$. We denote the class sums in $\mathbb{C}G$ by $C_i = \sum_{g \in C_i} g$. These class sums span the centre of the group algebra, they multiply according to the formula

$$C_i C_j = \sum_k a_{i,j}^k C_k$$

with $a_{i,j}^k \in \mathbb{Z}$ (We have shown this in the lecture).
Show that

$$a_{i,j}^k = \frac{|C_i|\,|C_j|}{|G|} \sum_{\chi \in \mathrm{Irr}(G)} \frac{\chi(g_i)\chi(g_j)\overline{\chi(g_k)}}{\chi(1)}$$

(Hint: use the second orthogonality relation and central characters.)

29) Let G be a finite group. Show:
a) If $g \in G$ and $x \in G$. There is a $y \in G$ such that g is conjugate to $[x, y] = x^{-1}y^{-1}xy$ if and only if

$$\sum_{\chi \in \mathrm{Irr}(G)} \frac{|\chi(x)|^2\,\overline{\chi(g)}}{\chi(1)} \neq 0$$

Hint: Assume that $g \in C_g$, $x \in C_x$ and $x^{-1} \in C_{x^{-1}}$ (conjugacy classes) and show that g is conjugate to $[x, y] = x^{-1}x^y$ if and only if $\mathcal{C}_x \cdot \mathcal{C}_{x^{-1}}$ has a nonzero coefficient for $\mathcal{C}_g$. Then use central characters.
b) g is a commutator (i.e. there exist $a, b \in G$ such that $g = [a, b]$ if and only if

$$\sum_{\chi \in \mathrm{Irr}(G)} \frac{\chi(g)}{\chi(1)} \neq 0$$

Bonus Problem) For a finite group G we define $G' = \langle [a,b] = a^{-1}b^{-1}ab \mid a,b \in G \rangle \lhd G$ (the *derived subgroup*, it also is the smallest normal subgroup with abelian quotient). It is generated by commutators, but, as we will see, not every element is required to be a commutator.
Let $G = \langle (1,2,5)(3,6,11)(4,7,9)(8,10,12), (1,8,4,3)(5,12)(6,10)(9,11) \rangle$ be a certain group of order 96. Then (e.g. using GAP) we obtain the character table of G as given below.
a) Determine in this table the classes which constitute G'.
b) Show using the criterion of problem 29b) that there is an element in G' that is not a commutator (but only a product of commutators).
c) If G is a simple group then clearly $G = G'$. The conjecture of Ore – recently being proven by Liebeck, O'Brien, Shalev and Tiep – states that for a simple group G all elements are proper commutators. Verify this conjecture for A_6, using a character table (obtained from GAP, or whatever source you deem appropriate).

2	5	1	4	4	5	1	1	4	4	5	1	5
3	1	1	.	.	.	1	1	.	.	.	1	1
	$1a$	$3a$	$4a$	$4b$	$2a$	$3b$	$6a$	$4c$	$4d$	$2b$	$6b$	$2c$
χ_1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	α	1	1	1	$\overline{\alpha}$	α	1	1	1	$\overline{\alpha}$	1
χ_3	1	$\overline{\alpha}$	1	1	1	α	$\overline{\alpha}$	1	1	1	α	1
χ_4	2	-1	.	.	-2	-1	1	.	.	2	1	-2
χ_5	2	$-\overline{\alpha}$	.	.	-2	$-\alpha$	$\overline{\alpha}$	.	.	2	α	-2
χ_6	2	$-\alpha$	.	.	-2	$-\overline{\alpha}$	α	.	.	2	$\overline{\alpha}$	-2
χ_7	3	.	-1	-1	3	.	.	-1	-1	3	.	3
χ_8	3	.	β	$\overline{\beta}$	-1	.	.	1	1	-1	.	3
χ_9	3	.	$\overline{\beta}$	β	-1	.	.	1	1	-1	.	3
χ_{10}	3	.	1	1	-1	.	.	$\overline{\beta}$	β	-1	.	3
χ_{11}	3	.	1	1	-1	.	.	β	$\overline{\beta}$	-1	.	3
χ_{12}	6	.	.	.	2	.	.	.	.	-2	.	-6

with $\alpha = e^{\frac{4\pi i}{3}}$ and $\beta = -1 + 2i$.