22) Let V be a $\mathbb{C}G$-module with corresponding character χ. According to problem 15, we can decompose $V \otimes V = W_S \oplus W_A$. Let χ_S (respectively χ_A) the characters that belong to W_S (respectively W_A). Show that

$$\chi_S(g) = \frac{1}{2}\left(\chi(g)^2 + \chi(g^2)\right) \qquad \text{and} \qquad \chi_A(g) = \frac{1}{2}\left(\chi(g)^2 - \chi(g^2)\right)$$

23) Show that the row sums in a character table are nonnegative integers.
Hint: Consider the permutation character for the action of G on itself by conjugation.

24) Let G be a finite group of exponent e and ε a primitive e-th root of unity. (We know that all character values for G lie in $\mathbb{Q}(\varepsilon)$.)
Let $\sigma \in \mathrm{Gal}(\mathbb{Q}(\varepsilon)/\mathbb{Q})$ be a field automorphism. For a character χ of G we define a class function χ^σ by

$$\chi^\sigma\colon G \to \mathbb{C}, g \mapsto (\chi(g))^\sigma.$$

Then χ^σ is a character of G. (Proof: χ is afforded by a representation ρ, defined over a finite degree extension F of $\mathbb{Q}(\varepsilon)$. Extend σ to an automorphism of F (this can be done by a theorem from 567), then the representation ρ^σ, defined by applying σ to all matrix entries of $\rho(g)$ affords the character χ^σ.)

a) Let $g \in G$ with $|g| = m$. Show that $\chi(g) \in \mathbb{Q}$ for all $\chi \in \mathrm{Irr}(G)$ if and only if for every l with $\gcd(l, m) = 1$ the elements g and g^l are in the same conjugacy class.
b) Show that the character table of the symmetric group S_n contains only rational values. (As character values also are algebraic integers, this implies that the entries are in fact integers.)

25) The character table of the symmetric group S_5 is given as

	$1a$	$2a$	$3a$	$5a$	$2b$	$4a$	$6a$
χ_1	1	1	1	1	1	1	1
χ_2	1	1	1	1	−1	−1	−1
χ_3	6	−2	.	1	.	.	.
χ_4	4	.	1	−1	2	.	−1
χ_5	4	.	1	−1	−2	.	1
χ_6	5	1	−1	.	1	−1	1
χ_7	5	1	−1	.	−1	1	−1

where a dot (.) indicates a zero entry and the class names give the element orders, the letters are assigned to distinguish classes with the same element order.
a) Determine representatives for the conjugacy classes (for example from element orders and centralizer orders – the latter are obtainable via the 2nd orthogonality relation).

b) Write the permutation character χ_π for the natural permutation representation of S_5 as a sum of irreducible characters.
Hint: The GAP commands

```
c:=CharacterTable("S5"); mat:=List(Irr(c),i->List(i,j->j));
```

can be used to obtain a matrix with the character values so you do not need to solve linear equations by hand.
c) Let τ be the tensor product of χ_5 with χ_6. Write τ as a sum of irreducible characters.
d) Let ϕ be the permutation representation of S_5 on the cosets of A_4 (of degree 10). Write the corresponding permutation character as a sum of irreducible characters.
Hint: The coset A_4h remains fixed under g if $hgh^{-1} \in A_4$.
As the GAP command `RightTransversal` can be used to obtain a set of representatives for the cosets, the following command thus determines representatives of those cosets, which remain fixed under the group element g:

```
Filtered(RightTransversal(s5,a4),h->h*g/h in a4);
```