39) We want to show that induction is transitive, i.e. if $S \leq T \leq G$ and $\varphi$ is a representation of $S$, then $\varphi \uparrow^G_S$ is similar to $(\varphi \uparrow^T_S) \uparrow^G_T$. We will do so on the level of characters, i.e. show that if $\chi$ is the character of $\varphi$, then

$$\chi \uparrow^G_S = (\chi \uparrow^T_S) \uparrow^G_T$$

40) Let $G = A_4$ and $U = \langle (1, 2, 3) \rangle \leq G$. Consider the representation $\delta: U \to \mathbb{C}$, $(1, 2, 3)^a \mapsto \zeta^a$ where $\zeta$ is a primitive 3rd root of unity. Determine (for example as images for generators) the induced representation $\delta \uparrow^G_U$.

41) Let $G \triangleleft A$ and $\phi: G \to \text{GL}(V)$ a representation.

a) Let $a \in A$. We define $\phi^a: g \to (aga^{-1})^\phi$. Show that $\phi^a$ is a representation of $G$ which is irreducible if and only if $\phi$ is irreducible. (There is a general setup for reduction to normal subgroups that goes under the name of “Clifford Theory”.)

b) Now consider that $[A: G] = 2$. Show that for a given irreducible representation $\phi: G \to \text{GL}(V)$ either:

i) $\phi^a = \phi \forall a \in A$.

ii) There exists $b \in A$ such that $\phi \neq \phi^b$ and for any $a \in A$ we have that $\phi^a \in \{\phi, \phi^b\}$. (I.e. there exactly two conjugates of $\phi$.)

c) In case i), show that there must be exactly two irreducible representations $\gamma, \delta: A \to \text{GL}(V)$ such that $\gamma \downarrow_G = \phi = \delta \downarrow_G$.

d) Show that in case ii), there is an irreducible representation $\psi$ of $A$ such that $\psi \downarrow_G = \phi + \phi^b$.

42) Let $G$ be a finite group and $g \in G$.

a) Describe the irreducible characters of $\langle g \rangle$.

b) Suppose for every $m$ up to the maximal element order in $G$ the $m$-power map on the conjugacy classes of $G$, that is the map which for each class $C$ gives the class of its $p$-th powers is known. (This means that we know $\chi(g^m)$ for any $m$. Of course it is sufficient to only store it for $m$ prime.) Give a formula that gives the values for $\chi \uparrow^G_{\langle g \rangle}$ for all $\chi \in \text{Irr}(\langle g \rangle)$. (Such a formula is implemented in GAP by the command $\text{InducedCyclic}(c, "all");$)

43) Let $G$ be a finite group and $N \vartriangleleft G$, $\chi \in \text{Irr}(G)$ and $1_N$ the trivial character of $N$. Show: If $(1_N, \chi \downarrow_N) \neq 0$, then $N \leq \ker(\chi)$.