Mathematics 602 **Homework (due Feb 24)** A. Hulpke

8) a) Let $n = a + b$ with $1 \leq a < b$. Show that S_n acts primitively on the a-element subsets of n. (Hint: Assume there was a nontrivial block and construct a permutation that fixes one block element and maps another one out of the block.)
b) Show that $S_a \times S_b$ is a maximal subgroup of S_n.

9) Let $G \leq S_n$ be transitive with a unique block system with blocks of size $m|n$. Let $1 \in B$ be one block in this system
Show that $N_{S_n}(G) \leq N_{S_m}(\mathrm{Stab}_G(B)) \wr S_{n/m}$.

10) Let $G \leq S_n$ be transitive and let $N = N_{S_n}(G)$.
a) Show that there is a homomorphism $\psi\colon N \to \mathrm{Aut}(G)$ with $\ker\psi = C_{S_n}(G)$.
b) What is the image of ψ?
c) If G is cyclic, generated by an n-cycle, show that N has order $n \cdot \varphi(n)$ where φ is Euler's function.

11) Let $f(x) \in \mathbb{Q}[x]$ be monic and irreducible and K be the splitting field of f.
a) Show that there is a faithful homomorphism $\mathrm{Gal}(f) = \mathrm{Gal}(K/\mathbb{Q}) \to S_n$, given by the action on the roots of f. Let G be the image of $\mathrm{Gal}(f)$ under this map.
b) Show that G is imprimitive, if and only if there exist $g, h \in \mathbb{Q}[x]$,$\deg(g)$, $\deg(h) < \deg(f)$, such that $f(x) \mid g(h(x))$.
Hint: Both properties correspond to the existence of a proper subfield $\mathbb{Q} \leq S \leq \mathbb{Q}[\alpha]$, where $f(\alpha) = 0$. It might help to think of the most prominent case, namely $f(x) = g(h(x))$, for example x^6+x^3-1.

12) For a prime p, we denote by C_p the cyclic group of order p.
a) Show that the iterated wreath product

$$\underbrace{(\cdots((C_p \wr C_p) \wr C_p) \wr \cdots) \wr C_p}_{m\text{-fold product}}$$

(i.e. for $m = 1$, we get C_p, for $m = 2$ we get $C_p \wr C_p$, &c.) can be represented by permutations on p^m points, and show that its order is $p^{1+p+p^2+\cdots+p^{m-1}}$.
b) (Cauchy) For $n \in \mathbb{N}$, describe the structure of the p-Sylow subgroups of S_n.
Hint: Consider first the case that n is a pure power of p. Use problem 6.

13) a) Let G be a finite group, $N \lhd G$ and $F = G/N$. Let φ be the regular permutation representation of F. Show: G is isomorphic to a subgroup of $N \wr_\varphi F$. (i.e. F as a regular permutation group)
b) (GAP) Let $N = A_6$ and $F = C_2$. We want to determine the isomorphism types of groups G that have a normal subgroup A_6 with a factor group C_2:

1. Construct (using `WreathProduct`) the wreath product $w := A_6 \wr C_2$.

2. Let `u:=List(ConjugacyClassesSubgroups(w),Representative)` be a list of subgroups of w (up to conjugacy).

3. Select (using `Filtered`) those subgroups from U that have order 720 and a derived subgroup of index 2.

4. You will end up with 5 subgroups. Calculate(using `NrConjugacyClasses`) the number of conjugacy classes for each. This shows that only two of these groups could be isomorphic. Use `IsomorphismGroups` to verify this.

(You will end up with 4 nonisomorphic groups. Which of them is S_6? What does this tell about the automorphism groups $\mathrm{Aut}(A_6)$ and $\mathrm{Aut}(S_6)$?)