Mathematics 601 **Homework (due Oct 15)** A. Hulpke

1) For a prime p, we denote by C_p the cyclic group of order p.
a) Show that the iterated wreath product

$$\underbrace{(\cdots((C_p \wr C_p) \wr C_p) \wr \cdots) \wr C_p}_{m\text{-fold product}}$$

(i.e. for $m = 1$, we get C_p, for $m = 2$ we get $C_p \wr C_p$ &c.) can be represented by permutations on p^m points and show that its order is $p^{1+p+p^2+\cdots+p^{m-1}}$.
b) (Cauchy) Show that for $n = p^m$ the largest power of p dividing $n!$ is $p^{1+p+p^2+\cdots+p^{m-1}}$. Conclude that in this case the m-fold iterated wreath product of C_p considered in a) is a p-Sylow subgroup of S_n.
c) Now consider a general $n \in \mathbb{N}$ and let $k = \log_p(n)$. Consider n written as a sum of p-powers

$$n = \sum_{i=0}^{k} a_i p^i \tag{1}$$

Show that the power of p that divides $n!$ is exactly

$$p^{\left(\sum_{i=1}^{k} a_i \cdot \left(\sum_{j=0}^{i-1} p^j\right)\right)}$$

d) By dividing the points $\{1, \ldots, n\}$ into units of p-power length according to equation (1), describe the structure of a p-Sylow subgroup of S_n.

2) For a prime power q consider the group $GL_n(q)$ consisting of all $n \times n$ matrices with nonzero determinant over the field $\mathbb{F}_q$ with q elements. This group acts naturally on the $q^n - 1$ nonzero vectors of the space $\mathbb{F}_q^n$.
a) Show that this action is imprimitive with blocks of size $q - 1$, corresponding to 1-dimensional subspaces of $\mathbb{F}_q^n$.
b) Show that the action on the $\frac{q^n-1}{q-1}$ 1-dimensional subspaces is primitive.
Note: This action on the 1-dimensional subspaces has scalar matrices (which form the center of GL) in the kernel. The acting group is thus $PGL_n(q) = GL_n(q)/Z(GL_n(q))$, called the "projective general linear group". Similarly there is PSL which also acts primitively on the 1-dimensional subspaces.

3) For each of the five major classes of the O'Nan-Scott theorem give the smallest two degrees in which a primitive group of this class arises and (for at least one of the degrees) give an example of such a group. You might find the following list useful which you can use without proof:
This list gives all simple nonabelian groups of order < 1000 and their maximal subgroups.

- $A_5 \cong PSL_2(4) \cong PSL_2(5)$ of order 60, maximal subgroups of index 5,6 and 10.
- $PSL_2(7) \cong PSL_3(2)$ of order 168, has maximal subgroups of index 7 (two classes!) and index 8.
- $A_6 \cong PSL_2(9)$ of order 360, has maximal subgroups of index 6 (two classes!), 10, 15 (two classes!)
- $PSL_2(8)$ of order 504, maximal subgroups of index 9, 28, 36.
- $PSL_2(11)$ of order 660, maximal subgroups of index 11 (two classes!), 12, 55.

Together with A_7, A_8, A_9 and A_{10} this includes all simple groups that have a primitive action of degree ≤ 10.

4) The O'Nan-Scott theorem gives us that every permutation group is contained in one of the maximal groups in each class, but it does not state anthing of inclusion of one class in another. Can you describe for any of the classes when (and under which condition) the groups in a class are maximal in S_n?